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Plato

A fanciful portrait. From a drawing by Raphael in the Academia at Venice. Inserted to show this artist's conception of the philosopher.

THE MATHEMATICS TEACHER

Volume XXIII No. 4

Edited by William David Reeve

Geometry in the Junior High School

By MARIE GUGLE
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WHEN THIS topic was first suggested, it was limited to geometry in grades seven and eight, with the questions appended, "What should be taught?" and "How should it be taught?"

Perhaps the best treatment of the topic would be to answer some such questions as the following:

Why should any geometry be taught in the junior high school?

What geometry should be taught?

Where in the junior high school course should it be taught?

And, How should it be taught?

In the brief time allotted, any answers must be given more or less categorically. Others may have answers as good or better, but, as Dr. Bode said recently, "On this occasion, my answers are official."

In one of Dr. Bode's philosophy classes, when in the discussion the students were pressing him rather hard about some of his views, he re-stated his opinions and said, "Well, in this class, at least, these conclusions are official."

Let us consider the first question:

Why should geometry be taught in the junior high school?

Many persons object to the introduction of any geometry in the seventh and eighth grades because, as they say, it must be done at the expense of arithmetic and pupils know little enough arithmetic as it is.

Arithmetic has been one of the three R's so long that everybody accepts it as one of the "fundamentals" of the elementary school. It is recognized that everyone has to do some "figuring"; therefore, everybody should have reasonable skill in the so-called fundamental processes with integers and simple fractions in both common and decimal form, with, perhaps, a little percentage.

It should be remembered that during the middle decades of the nineteenth century, the arithmetic classes were filled, not with children but with adult pupils, whose more mature interest was whetted by the introduction of the puzzle type or catch problem and by the more difficult topics, such as, alligation, cube root, and partial payments. These and other similar topics were introduced solely because they appealed to the maturer interest of these over-age but not subnormal pupils. No one ever claimed that there was a universal need for these topics or that they were practical.

Gradually the school year was lengthened from three months to six, eight, and ten months. A sixth grade child, today, eleven years of age, has spent sixty months in school, whereas his great grandfather, up to the age of twenty-one, had spent only forty-five months or less in school. Dr. Judd says that the present sixth grade pupil has read vastly more books than his great grandfather did in his whole lifetime. Nevertheless, he is still immature and lacking the background of experience to such an extent that those topics of arithmetic introduced for the benefit of the adult pupil are no longer appropriate.

It has become necessary, therefore, to challenge the various topics of arithmetic, to find out why they should or should not be taught. As a result, many of the older topics have been replaced by others of more vital interest. Yet the traditional idea remains that arithmetic is taught because it is so generally useful. By such a test, the one application of arithmetic that is most universally needed is that of making budgets and keeping simple accounts. Lack of thrift is said to be one of our national vices. Schools are now required by law to teach thrift. Such teaching means little unless it is translated into a habit of thrift and the habit of thrift is best developed by keeping personal accounts.

In the Second Yearbook of the National Council of Teachers of Mathematics, Mr. C. L. Thiele of Detroit shows the general social needs in the field of arithmetic. He says: "A survey of the actual mathematical needs will show that a thorough mastery of percentage,

decimal fractions, common fractions, graphing and charting, ratio and proportion, as well as some practice in elementary accounting, would be needed to meet the home budget problem of the consumer."

We see, therefore, that the introduction of other phases of mathematics into the junior high school is not done at the expense of arithmetic. The increased number of days the pupil attends school and the elimination of unneeded materials leave time for other kinds of mathematics.

Now, we are ready to consider more specifically, why some of this added material should be geometry. For an authoritative answer let us consult the report of the National Committee which is the Bible of the mathematics teachers and it is always helpful to read the Bible. On page 7, among the practical aims to be served by the study of mathematics are "familiarity with geometric forms common in nature, industry, and life; the elementary properties and relations of these forms including their mensuration; the development of space perception and the exercise of spatial imagination." On page 9, among the disciplinary aims is listed "the acquisition, in precise form, of those ideas or concepts in terms of which the quantitative thinking of the world is done. Among these may be mentioned ratio and measurement (lengths, areas, volumes, weights, velocities, and rates in general), proportionality and similarity, positive and negative numbers, and the dependence of one quantity upon another." On page 20, it says: "In the years under consideration (grades seven, eight, and nine), it is especially important to give pupils as broad an outlook over the various fields of mathematics as is consistent with sound scholarship."

As a matter of fact our much vaunted arithmetic has always borrowed a great deal from geometry. It would be very difficult to teach a pure arithmetic, one from which all geometric phases had been eliminated. Such arithmetic would include the abstract process of counting (although the sticks, circles, and other counters used and the concrete objects counted are geometric); it would include only the abstract calculations, for all measures would have to be excluded as geometric. The old arithmetic borrowed much from geometry, organized it into a formal, meaningless, and uninteresting type of mensuration problems, solved by rule of thumb. Such material lent itself to the older methods of teaching. Instruction was mechanized. It was mere performance of tasks; getting the answer was the goal.

With the introduction of methods based on the newer psychology,

it became necessary for the child to see meanings in the materials and processes handled; to be an active learner, not a passive recipient; to learn through purposeful activity.

The intuitive geometry introduced into the junior high school is an attempt to put new meanings into the old mensuration; to enlarge the child's horizon so that he may get a complete, though limited, survey of the whole field of elementary mathematics.

Since there are many pupils for whom the junior high school means the end of their schooling or at least the end of their study of mathematics, it is far better to let them see the whole of the elementary field instead of limiting their vision to one small part. That is the only way in which they can get any true perspective and see any meaning to mathematics. To illustrate, I have often used the figure of a picture. If I showed you a large canvas on which an artist had painted a masterpiece, you would understand the meaning of its composition, see the relationship between the various parts, appreciate the interplay of lights and shadows; in other words you would understand, appreciate, and enjoy the picture. But, suppose I should cut out a piece of canvas from the center or other part of this picture. You would say, "What is this? What does it mean? I see some paint and bits of color, but a masterpiece! No!" Yet it really is a part of the original masterpiece, worthless because it is a fragment. If, on the other hand, I should show you a reproduction, in miniature, of the entire picture, while it might lack some of the beauty and impressiveness of the original, you would still understand its composition, see its meaning, appreciate its color, and react to its beauty. So it is with mathematics. If we wish pupils to see a meaning in it and to appreciate any of its real usefulness and beauty, we must take off the blinders that have limited their vision to one phase of the subject and enlarge their vision to the whole field. It is better to see a limited distance in all directions than to see a little farther in one direction only.

In the National Council's First Yearbook, page 162, Mr. Betz has made an excellent statement of the reasons why arithmetic and geometry jointly are the foundations of mathematics. He says:

> Historically, mathematics was called into existence because of the necessity for counting and measuring. This fact has given mathematics a double foundation, namely arithmetic and geometry. It is apparent that we cannot make or manufacture the simplest article without giving due attention to its form,

its dimensions, and the proper relation of its parts. Nature and the manual arts are readily seen to be the two permanent sources of geometry. More especially, training in space intuition and in plastic thinking is at the bottom of all forms of applied art. The geometric principles of equality, symmetry, congruence, and similarity are implanted in the very nature of things. The art of measurement permeates the fabric of modern civilization at every point. Then, too, intuitive geometry has a functional aspect through the unique training which it affords in the discovery and formulation of relationships. Finally, intuitive geometry is absolutely essential as a preparation for effective work in demonstrative geometry.

When thus conceived, intuitive geometry serves to vitalize and humanize the whole course in elementary mathematics.

When formalism and task performance were the method and end of instruction, it was quite natural to have a formal and meaningless algebra follow the formal arithmetic. Modern psychology has demanded that algebra teaching also be re-vitalized and experience has shown that nothing else does that so well as intuitive geometry. Therefore, we see that the double foundation of arithmetic and intuitive geometry is as necessary for algebra as for demonstrative geometry.

Summarized briefly, some reasons why intuitive geometry should be taught in the junior high schools are as follows:

1. Increased attendance at school has given more time for arithmetic in the first six grades than was formerly had by pupils up to the age of twenty-one.

2. The lack of maturity of pupils today has necessitated the elimination of puzzle type or catch problem and other phases of arithmetic which require greater maturity in the pupil for reasonable understanding.

3. It has been found that with the changing social and economic structure of our society, many newer phases of arithmetic should be substituted for obsolete topics.

4. To be consistent with the broad outlook and exploratory objectives of the junior high school, an opportunity for a complete survey of the field of elementary mathematics should be offered.

5. Intuitive geometry in the junior high school is a means of following the demands of the new psychology that "motor activity is a basic condition of intellectual growth." It deals with concrete things and becomes the basis of much "purposeful pupil activity."

6. It is such an integral part of nature, the manual arts, and esthetic art that its understanding is necessary in order that the pupil be able to adjust himself to his environment.

7. Intuitive geometry is a necessary foundation to and preliminary preparation for the study of algebra and demonstrative geometry, not to mention trigonometry.

For these reasons and others that might be given, we should agree that intuitive geometry has a necessary and rightful place in the mathematics offered to all pupils in the junior high school.

Our second question is:

What Geometry Should Be Taught?

For an answer to this question, let us turn again to our Bible, the National Report. On page 22, it suggests the following:

(a) Direct measurement of distances and angles by means of linear scale and protractor. The approximate character of measurement. An understanding of what is meant by the degree of precision as expressed by the number of "significant" figures.

(b) Areas of the square, rectangle, parallelogram, triangle, and trapezoid; circumference and area of a circle; surfaces and volumes of solids of corresponding importance; the construction of corresponding formulas.

(c) Practice in numerical computation with due regard to the number of figures used or retained.

(d) Indirect measurement by means of drawings to scale; use of square ruled paper.

(e) Geometry of appreciation; geometric forms in nature, architecture, manufacture, and industry.

(f) Simple geometric constructions with ruler and compasses, T-square, and triangle, such as that of the perpendicular bisector, the bisector of an angle, and parallel lines.

(g) Familiarity with such forms as the equilateral triangle, the 30°-60° right triangle, and the isosceles right triangle; symmetry; a knowledge of such facts as those concerning the sum of the angles of a triangle and the pythagorean relation; simple cases of geometric loci in the plane and in space.

(h) Informal introduction of the idea of similarity.

Such a list at first impresses one as being as formal as the condemned mensuration of the old arithmetics. Any such condensed classification will appear so because it is a formal classification. But what should be taught is not as important as the way it should be taught. Therefore, in answer to this question, I am submitting below,

a list of geometric ideas or concepts and relationships that can be used effectively in the junior high schools. Such a foundation will be a helpful basis for an increased understanding of form and space in nature, in science, in industry, and art, and in possible later study of algebra, geometry, and trigonometry.

OUTLINE OF THE GEOMETRY WHICH SHOULD BE TAUGHT IN THE JUNIOR HIGH SCHOOL

It is surprising how many geometric ideas and geometric facts or relations can be developed experimentally or intuitively in the junior high school course. Since the material is used to vitalize and illuminate the arithmetic and algebra, it is given, not at a sacrifice of instruction in arithmetic and algebra, but as a means of increasing and improving such instruction. The following list is in no sense complete or limited in kind or sequence. It merely shows the geometry that can be taught and is being taught effectively in many junior high schools.

I. Geometric Tools and Their Use

Ruler and steel tape
Compass and protractor
Triangles: 45 degree and 30-60 degree right triangles
Carpenter's square, T-square
Meter sticks and metric units
Squared paper and quadrant
Symbols for perpendicular, parallel, angle, triangle, square, rectangle, etc.

II. Geometric Construction

Bisection of lines and angles
Drawing perpendicular lines
Construction of square, rectangle, triangles of all kinds, hexagon, octagon, pentagon
Construction of parallel lines
Right, acute, and obtuse angles inscribed in circles
Construction of various kinds of figures equal to given figures
Dividing line into equal parts
Drawing to scale
Use of various forms in design

III. Geometric Concepts*

A. *Geometric Forms*

Geometric forms in nature, art, industry, manufacture, construction

B. *Angles*

Angles and their measurement
Adjacent angles, opposite angles

*The thorough understanding of a geometric concept does not necessarily mean the ability to formulate a definition.

Complementary angles
Exterior and interior angles
Vertex
Vertical, supplementary, and corresponding angles
Alternate vertices

C. *Solids*

Rectangular parallelopiped, cube, lateral and total surfaces
Prism
Cylinder, cylindrical surface
Cone
Sphere, spherical surface
Pyramid
Volume, capacity
Cubic inch, cubic centimeter, etc.
Base, height

D. *Plane Figures*

Lines: Straight, curved, horizontal, vertical, oblique, parallel, perpendicular, edge, direction, pendulum, plumb line, intersect, intersection, distance, transversal, bisect, bisector, perimeter

Triangles: equilateral, isosceles, scalene, right, equiangular, base, altitude, hypotenuse, gable end

Quadrilaterals: square, rectangle, rhombus, parallelogram, trapezoid, diagonal, surface, area, base, altitude, square inch, square centimeter, etc.

Polygons: Number of sides, regular, inscribed

Circle: radius, diameter, circumference, area, arc, concentric circles, pi

E. *Miscellaneous*

Ratio, graphs of ratio and percents
Gravity
Trigonometric functions, represented geometrically
Theorem and proof

IV. Geometric Abilities and Relationships

How to make and to interpret graphs
How to mark and to read angles and polygons
How to measure lines on squared paper, with ruler, with compasses or dividers

Formulas:

(a) Perimeters of square, rectangle, circumference of circle
(b) Areas of square, rectangle, triangle, parallelogram, trapezoid, circle, cylindrical surface, lateral surface
(c) Volumes of cube, rectangular prism, cylinder

Sum of the angles of a triangle equals 180 degrees.

A perpendicular is the shortest line that can be drawn from a point to a line.

At a point in a line or from a point to a line, only one perpendicular can be drawn.

Properties and uses of right triangle
Pythagorean theorem and uses
Uses of parallel lines
Diagonals of a rectangle are equal and bisect each other.
Relations between opposite sides of square, of rectangle, of rhombus, and of parallelogram
Relations between the diagonals of these quadrilaterals
How the diagonals of these figures divide the angles at the vertices
The relation between the triangles made in these figures by their diagonals
The relation between the diagonal and side of a square
Similar figures and their uses: rectangles, parallelograms, triangles
Drawing to scale: interpretation of maps, blue prints, etc.
Proportion
Finding heights from shadows
Measuring inaccessible distances
Sum of angles of polygons
Demonstration of six or eight propositions
Symmetry

Our third question is:

Where in the Junior High School Should Intuitive Geometry Be Taught?

The placement of topics should follow one basic principle. Since a great many pupils, for economic reasons or otherwise, drop out of public school as soon as they reach the end of the compulsory school age, we find pupils leaving school in increasing numbers at the end of each year of the junior high school, and somewhat during the term itself. Therefore, all vital phases or units of elementary mathematics should be listed and rated as to their importance not for their use in later study but for the universality of their need in the workaday world. The one basic principle of placement, then, should be to put the most generally needed phases earliest or in the seventh grade, assuming, of course, that no unit beyond the experience and ability of the pupils is to be included at all.

By following such a principle, the work of the seventh year will be more largely arithmetic, covering the common social and business phases, including budgets and simple accounts. Throughout the junior high school there should be continued practice in the fundamental processes, both incidental to other work and in definitely planned drill. In connection with the arithmetic, the making and reading of line and bar graphs may be effectively introduced; also the simplest ideas of measurement, in both English and metric units

and on square ruled paper. Approximations in all measurements should be brought out clearly. The simple mensuration formulas may be introduced here naturally and effectively.

Pupils in the seventh grade can learn and do enjoy more intuitive geometry. For that reason some schools have introduced much of it in the seventh grade. However, I fully believe that the other principle, namely that the most generally needed phases of mathematics should come earliest in the course, is a safer one to follow. Therefore, in the seventh grade everyday uses of arithmetic, budgets, simple accounts, and thrift should take precedence over a large amount of intuitive geometry.

By following this same principle of selection, the work of the eighth grade will be largely intuitive geometry, but in it there is a constant use of arithmetical computation, and through it there is a meaningful introduction of general number and the fundamental algebraic processes with positive expressions.

By this principle of selection, the work of the ninth grade will be more largely algebraic, but constantly illuminated by geometry and graphs and continuing arithmetical computation in checking. If, in the junior high school, the pupil is to survey the whole field of elementary mathematics, he should have a glimpse into numerical trigonometry and into demonstrative geometry. The latter might well be only enough to show the pupil what a demonstration is, in order that he may know whether or not he wishes to elect the subject in the senior high school. It is through trigonometry, however, that he gets his most attractive long-distance vision of mathematics.

Our fourth question is:

How Should Intuitive Geometry Be Taught?

Unfortunately, no one can answer that question specifically and finally. I said "unfortunately," but it may be "fortunately" in the long run. If some one could prescribe *the* best method for teaching a given topic or unit of instruction, teachers would cease to be creative artists and become automatic purveyors of instruction. The "how" of teaching is always a variable and for that reason it is the challenge that keeps our profession alive and growing. The best "how" for one pupil or class is not the best for another. There is no specific that can be prescribed as a cure-all for our ailments in the teaching of mathematics.

Experience has shown us that some methods are ineffective and waste time; that following certain principles usually leads to the best results.

It is possible that the suggested intuitive geometry may be taught in as formal a manner as the mensuration of the old arithmetics. The usual formal procedure is about as follows:

1. Definition of terms.
2. Classification and further definition of various kinds.
3. Formal statement of relationship as a rule to be memorized and followed.
4. Example showing the use of rule.
5. Set of exercises for solution.

Such a procedure sets no objective or goal, offers no motive or stimulus, involves no activity necessary for real learning, and leads to no genuine adaptation. Such a method continues the lesson learning procedure and the performance of an assigned task, so thoroughly condemned by H. C. Morrison in his book, The Practice of Teaching in the Secondary Schools.

A more acceptable general procedure might be somewhat as follows:

1. The teacher gives some sort of test to find out what the pupil knows about the unit.
2. He surveys the pupil's apperceptive background to find whether or not it is adequate. If inadequate, he supplies the necessary background.
3. In presenting new material, his first duty is to give the pupil such an appreciation of the unit that he really feels the need of learning and the desire to learn it.
4. Then he directs the pupil to some activity, through which the pupil will learn to think about the unit. In intuitive geometry the pupil may gather material, make objects, draw figures, cut or fold paper, etc. Then he is led to observe, test, measure, compare, form judgments, and verify them. After sufficient exploration and experimentation, he is led to summarize and generalize.

In this thinking process the pupil will be introduced to new terms only as needed names for concepts already developed and understood. He will develop his own definitions and rules as summaries and conclusions, instead of having them given to him ready made at the beginning of the unit.

Once, when Dr. Denny of Ohio State University was addressing a

group of teachers of English, he used this epigram: "A definition is meaningless until it is useless."

5. During this directed study, the class period is a laboratory period and the classroom, a work shop, in which the individual pupil is learning through his own self directed activity, under the guidance of the teacher.

6. When the pupil has completed his project and feels he has mastered the unit, he should take a test designed to show whether or not he has mastered it. If he has, he should be assigned other special projects or readings. If he has not mastered it, the teacher should re-teach it and give him more material to work with.

7. When all the members of the class have satisfactorily completed the unit, a class discussion should follow, reviewing the unit as an organized whole, stating the pupils' appraisal of its meaning, importance, and usefulness, and establishing and extending their appreciation of it through its applications.

Perhaps it is futile, if not presumptuous, for me to attempt to answer this question of how, since you have available such books as Morrison's, The Passing of the Recitation by V. T. Thayer, and others.

Thayer says that "this distinction between matter and method rests, however, upon the passive conception of learning"; that "learning as an active process involves an intimate identification of the what and the how." He says further:

> The result is an emphasis upon what Professor Counts calls procedure values which was absent in early teaching. Subjects of study, such as history, science, mathematics, language, and the like, are now being organized for teaching purposes in terms of definite pupil activities: and these pupil activities are specifically designed to develop methods of thinking, habits of procedure, peculiar to these subjects and essential for independent progress in them.

Limited time forbids any discussion of Morrison's mastery plan through units of instruction.

Practically all writers agree that before any real learning can take place, the pupil must have an adequate background of experience or an apperceptive mass. Perhaps, the main objective in teaching intuitive geometry is to supply the child with this background. Therefore, he must see, handle, and explore things for himself. Otherwise he will not be getting experience. He should learn things, not about things in his surroundings.

Much emphasis is now being laid by psychologists on "the motor

set, the original impulse, the drive or purpose," as a first phase of learning of peculiar importance; for "no learning will take place unless there is an initial drive in the subject; a drive sufficient not only to start an action but to see it though in spite of difficulties."

Before teaching a new unit, therefore, it is very necessary for the teacher to have the pupil feel a need for learning the unit. A mere statement that he will need to use it in some future study will not suffice.

Since learning is now conceived as developed only through "purposeful activity," it is necessary to organize the unit into a project involving such activity, for mere lesson learning or problem solving is not education. Psychologists agree that "the intellect cannot be educated in isolation; and that physical activity and emotional elements are vital factors in growth."

As Dr. Thayer says:

> We thus see that the conception of learning as an activity controlled and directed both from within and without leads to educational methods which contrast with the early precepts of teaching. It places in the foreground an appeal to the genuine interests of children as starting points for instruction and it defines the outcomes of education in terms of interests, i.e., dynamic ideals and habits. It recognizes that curricular materials cannot center exclusively upon adult value. Only in the schoolroom have we expected human beings to work industriously at something the worth of which must be taken wholly on faith.—Mental and moral decay are inevitable outcomes when one's daily occupation reduces itself literally to effort for effort's sake.

Mathematics textbooks are full of problems and necessarily so. However, too often the end and aim of mathematics teaching has been the mere ability to solve problems, sometimes the harder the better. Year in and year out, from the earliest elementary grades up, pupils solve problems and more problems, seldom seeing what they are all about.

These problems are called by Morrison the assimilative material, the experience basis, out of which the real learning may arise. The means is sometimes confused with the end. No doubt, you have more than once asked a pupil whether or not he has studied simple equations, or quadratic equations, or the like, only to have him reply: "I don't know, but if I had a book I could show you what page we are on."

Mathematics is said to be a method of thinking. But how often have we expected the child to think without giving him anything to

think about or any motive for thinking? Through the intuitive geometry he is supplied with material, interesting, concrete, within his comprehension, about which he can learn to think mathematically. But we must abandon our old tasks-assigned, problems-to-be-solved, ground-to-be-covered methods. We must select various phases or units of mathematics and be able to show the pupil why he needs to know them. We should show him that there are several phases of mathematics. One is computation and skill in manipulation. Ability in this phase comes from right practice continued to 100 per cent correctness to be worth anything. Another phase is a thorough understanding of the underlying principles, to be mastered as completely as he mastered the art of reading, a permanent adaptation, always ready for use when needed. A third phase is an appreciation of mathematics, as a tool, as a mode of thought, as a method of reasoning, as a basic element in the progress of civilization and science, and as a foundation for all that is beautiful in art and nature. To say how this should be done best is well nigh impossible. But it is an interesting task for every mathematics teacher and a big challenge.

The following sample lesson plan prepared by C. B. and Helen Marquand of Columbus, may be of interest here:

SAMPLE LESSON PLAN*

INTUITIVE GEOMETRY

Unit on Triangles

A. Teacher's Knowledge
   1. Significance of intuitive versus demonstrative geometry
      a. History of each
      b. Psychology involved
      c. Relation to curriculum organization
   2. Early Knowledge
      a. Babylonians
      b. Chinese
      c. Romans
   3. Congruence and similarity
      a. Congruence shown experimentally
      b. Similarity as seen in common objects
         (1) Photography
         (2) Models
         (3) Maps and plans

* Adapted from "A Course for Teachers of Junior High School Mathematics" by W. L. Schaaf. Bureau of Publications, Teachers College, Columbia University.

4. Symmetry
   a. Familiar objects
   b. Nature
   c. As shown experimentally
   d. As used in pattern and design

B. Teacher's Objectives
   1. Information
      a. Origin of Geometry
         (1) Knowledge of forms
         (2) Early shelters
         (3) Farming and surveying
      b. Geometry of form, size, and position
         (1) Shapes of figures
            (a) Regularity
            (b) Symmetry
            (c) Similarity
         (2) Size of figures
            (a) Method of measuring length
            (b) Measuring angles
            (c) Measuring areas
   2. Skills, habits and concepts
      a. Knowledge of fundamental geometrical concepts
      b. Knowledge of terminology
      c. Knowledge of notation and truths
      d. To train hands in use of tools
      e. To train the eye to see geometrical relationships
      f. To train the mind in the methods of studying geometrical data by the inductive process
      g. To give the training in desirable habits of thought, achievement, and conduct

C. Appreciation Background
   1. Symmetry in nature
      a. Inanimate objects
         (1) Snowflakes
         (2) Crystals
      b. Living forms
         (1) Leaves, trees, flowers
         (2) Human beings
         (3) Cross section of fruits
   2. Application of geometry to art
      a. Painting and drawing
         (1) Perspective
      b. Design
         (1) Patterns
         (2) Friezes
   3. Application of geometry to architecture
      a. Laws of art

(1) Balance
(2) Monotony and diversity
(3) Rhythm

b. Illustrations
(1) Bridges
(2) Cathedrals
(3) Monuments
(4) Modern buildings

4. Dynamic Symmetry
a. Uses
(1) Art (Greek and Modern)
(2) Design
b. In nature
(1) Plants, flowers, etc.
(2) Dragon Fly

D. Educational Values
1. Exploratory function
a. To open the door to geometric knowledge
b. To develop awareness of the universe of space
c. To acquaint pupils with the nature of geometry
2. Significance of intuition
a. Encouragement of experimentation
b. Realization of intuition, belief
c. Development of appreciation of need of verification

E. Psychological Considerations
1. Use life situations
a. Genuine problems
b. Pupils find real need
2. Learn by doing
a. Manipulation of tools such as straight edge ruler, compass, etc.
b. Experimentation with a variety of things
3. Pupil interests
a. Objects of nature and science
b. Practical applications
c. Puzzles and recreations
4. Inductive processes
a. Large number of observations
b. Observations of likenesses and differences
c. Formation of generalization
d. Verifying

F. Teaching Procedures
1. Pupil activity
a. Pre-test
(1) Familiarity with square and rectangle
(2) Familiarity with angles
b. Study of triangles from a square
(1) Cutting and folding on a diagonal

(2) Comparing triangles with the original square
(3) Reading angles
(4) Sum of the angles of a triangle
  (a) By calculating from a square
  (b) By placing the corners of triangle about a point on fold of paper
  (c) Same for general triangles
  (d) By use of protractor
(5) Development of related angles
  (a) Complementary angles
  (b) Interior angles
  (c) Exterior angles

c. Study of triangles from rectangles
  (1) Drawing several rectangles, cutting and folding on the diagonal
  (2) Study as for the square section-b

d. Development of formula for area
  (1) Comparison with formula for square and rectangle
  (2) How to find altitude
    (a) By folding
    (b) By protractor

e. Practice problems
  (1) Estimation
  (2) Computation from formulae

f. Construction of triangles
  (1) Right triangle with protractor
  (2) Triangles in circles
    (a) Right
    (b) Acute
    (c) Obtuse

g. Constructions of lines
  (1) Perpendicular lines
    (a) From a point within a line
    (b) From a point without the line
  (2) Lines of bisection
    (a) Bisect a line by folding
    (b) Bisect a line by using compass

h. Comparative study of lines in isosceles and scalene triangles
  (1) Bisectors of the vertex angles
  (2) Altitudes
  (3) Perpendicular bisectors of the bases

i. Designs
  (1) Using circles, triangles, squares, and rectangles
  (2) Finding and reporting uses of such designs as in linoleum, wall paper, textiles, mosaics, friezes, etc.

j. Study of the right triangle
  (1) Practical uses, such as bridges, brackets, woodwork, ironwork, cathedrals, building construction
  (2) Development of Pythagorean theorem

(a) Isosceles right triangle on squared paper
(b) Three-four-five right triangles
(c) Assumption on the general triangle
(d) Historical development
(e) Practical exercises
(f) Relationship of the diagonal of a square
(g) Meaning and geometric representation of the square root of 2 and 3

k. Appreciation through study of dynamic symmetry
(1) Basis of 90% of classic Greek design and architecture
(2) Basis of growth in plants and animals such as iris and dragon fly
(3) Use in modern art

l. Tests
(1) Minimum essentials
(2) Comprehension
(3) Information
(4) Vocabulary, etc.

m. Special projects

n. Socialized recitation
(1) Organization or summary of unit
(2) Pupil appraisal and adaptation

2. Suggestive helps

a. Points, lines, surfaces
(1) Use illustrations
(2) Use psychological order when possible
(3) Avoid introductions by formal definitions

b. Angles
(1) Varied and many illustrations
(2) Use practical suggestions

c. Instruments
(1) Ruler
(2) Compasses
(3) Protractor

d. Appreciation Lesson
(1) Mariner's compass
(2) Street maps
(3) Laying out streets and sub-divisions
(4) Slides and illustrations of dynamic symmetry

e. Geometrical construction
(1) Introduction by paper folding
(2) Follow by applying symmetry
(3) Give fundamental construction
(4) Introduce construction of various patterns, designs, etc.

f. Practical applications
(1) References—Histories of Mathematics
(2) Library readings and reports
(3) Samples of mathematical designs as used in practice

The Introduction to Demonstrative Geometry

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INTRODUCTION

DEMONSTRATIVE GEOMETRY presents to most pupils at once two great difficulties. The pupils are expected to acquire new ideas and to apply to them a method of argument that is formal and strange. They are asked to apply a new tool of thought, the geometric demonstration, to ideas that are highly abstract and often exceedingly vague.

Geometric demonstration involves the application of formal logic to definitions, postulates, and theorems.

Whatever belief, as mathematical philosophers, you hold concerning the nature of the foundations of geometry—that they are synthetic judgments a priori, experimental facts, or convenient conventions, as teachers of geometry to tenth grade pupils you know that these fundamental ideas are difficult, have to be formulated, need illustration with familiar objects and concepts, and are mastered only after long, carefully directed use. We no longer define axioms as self-evident truths but often teach them as if they were. Pupils learn to use the theorems about congruent triangles accurately and fairly readily long before they can identify the "same thing" in the axiom "Things equal to the same thing are equal to each other." More care is usually taken in the text and in the teaching to illustrate and apply the theorems than the axiom, probably because the mere statement of the axiom is supposed to make its truth break forth with a blazing and clarifying light. The "self-evident" myth still affects teaching.

We teach demonstrative geometry so that high school pupils may learn the methods of geometric proof. The most significant cause of early discouragement and later failures is a lack of adequate introduction to geometric notions through concrete experience and intuition before beginning formal proofs. In presenting arguments for more attention to introductory work I shall speak about earlier recommendations on this topic, practice in other countries, our present practice, and possible improvements.

Earlier Recommendations

The Report of the Committee of Ten on Secondary School Studies contains the report of a Conference on Mathematics headed by Newcomb and Byerly. The report of this Conference is dated March, 1893, thirty-seven years ago. I have read the report more than once and have wondered how long after my time it will be until American high schools shall have moved up to it.

The Conference recommends that the child's geometrical education should begin as early as possible; in the kindergarten, if he attends a kindergarten, or if not, in the primary school. He should at first gain familiarity through the senses with simple geometrical figures and forms, plane and solid; should handle, draw, measure, and model them; and should gradually learn some of their simpler properties and relations. It is the opinion of the Conference that in the early years of the primary school this work could be done in connection with the regular courses in drawing and modeling without requiring any important modification of the school curriculum.

At about the age of ten for the average child, systematic instruction in concrete or experimental geometry should begin, and should occupy about one school hour per week for at least three years. During this period the main facts of plane and solid geometry should be taught, not as an exercise in logical deduction and exact demonstration, but in as concrete and objective a form as possible.

Of course, while no attempt should be made to build up a complete logical system of geometry, the child should be thoroughly convinced of the correctness of his constructions and the truth of his propositions by abundant concrete illustrations and by frequent experimental tests; and from the beginning of the systematic work he should be encouraged to draw easy inferences, and to follow short chains of reasoning.

It is the belief of the Conference that the course here suggested, if skillfully taught, will not only be of great educational value to all children, but will also be a most desirable preparation for later mathematical work.

The Report of the National Committee of Fifteen on Geometry Syllabus, dated July, 1912, discusses in Section C preliminary Courses for Graded Schools. "It is of the utmost importance that some work in geometry be done in the graded schools." This Report points out the value of drawing, drawing to scale, pattern making, mensuration, discovery of theorems as facts, and construction with ruler and compasses in developing and fixing ideas preparatory to demonstration.

That such work is of vital value to the student can scarcely be doubted; that it is absolutely legitimate will probably be admitted by all interested in primary education. Its value, its real direct motives, its contact with life, the legitimacy of its subject-matter exceed incomparably those of the traditional

course in advanced arithmetic. At least, the course in arithmetic may be vitalized by a liberal infusion of such geometric work.

If such a course is not given in the grades—perhaps even though it is—a course of similar character but very much shorter may be given in the high school before formal work in demonstrative geometry is attempted. In any event it is desirable that the course in formal geometry should not proceed in its traditional groove until the teacher is assured that the ideas mentioned above are thoroughly familiar to the student.

The Report of the National Committee on Mathematical Requirements in Secondary Education repeatedly emphasizes the importance of intuitive geometry, states its purposes, and outlines a course. I need not repeat here the recommendations of that well-known Report.

Practice in Other Countries

The recommendations of all of these reports are in accord with the practices in other countries.

England. The most valuable discussion that I know on the teaching of geometry is the Report on the Teaching of Geometry in Schools prepared for the British Mathematical Association and published in 1923. Instruction in geometry is here divided into five stages.

Stage A. The Experimental Stage. This is largely the solution by drawing, measurement, and computation of problems found in the field or given in the class room. The aim is to give contact with geometric ideas, and to place them on a solid foundation of experience. The pupil's work is not mainly verbal but leads from experience to formal statements and introduces deduction. This stage ends at about the age of twelve and one-half years.

Stage B. The Deductive Stage. This is similar to the deductive work attempted in our tenth grade classes. It extends from the age of twelve and one-half to fifteen years.

Stage C. Systematizing Stage. From the ages of fifteen to sixteen or seventeen years.

Stage D. Modern geometry.

Stage E. Philosophy of geometry.

Germany. Propadeutik for Geometry has long been provided in the official announcements of courses in German schools. The following is taken from W. Lietzmann's *Methodik des Matematischen Unterrichts* I Teil (1926). He quotes from the government programmes of studies. The first quotation is from the *Lehrplan* of Württemberg for geometry in the secondary schools.

Preparation for the systematic teaching of geometry will be given in all classes of schools in *Klasse* III [seventh grade] through training in geometric intuition. This begins with the observation of simple solids. By this means will be developed the fundamental notions of geometry, the relations between lines and planes, and the notions of the simple geometric figures. Practice in drawing is given in connnection with this instruction, which from the beginning on desists from scientific definition. The pupils get instruction in the use of the straight edge, triangle, circle, graduated ruler, and protractor. They learn to make the elementary constructions by using these instruments, for example, erecting perpendiculars and bisecting angles and the construction of simple rectilinear figures, and when abundant practice has given accuracy and facility, the simplest theorems of equality of angles, angle sums, and equality of surfaces are developed by empirical methods (translation, turning, superposition, measurement). In this connection there will be taken up the areas of figures bounded by straight lines whose lengths are determined by approximation or measurement. The approximate ratio of the circumference to the diameter will be found by the measurement of circular objects and from this will be developed the area of the circle.

Each pupil will cut from unruled paper each kind of triangle and quadrilateral. These figures with some disks (circles) of different sizes are to make a collection of geometric forms. The pupils will make sketches of the simple solids and from these make paper models.

This instruction shall gradually lead the pupils to the formation of geometric concepts and awaken in them the need for proof of the facts brought out by observation and intuition.

This kind of work in geometry is prescribed for all classes of schools. The following quotation is from the Prussian Programme of October 15, 1922, for the *Volkschulen*.

The instruction in geometry should train the pupil's geometric intuition and should develop in him the ability to comprehend, represent, estimate and compute the geometric forms that appear in life.

The materials to be considered for schools of one or of few classes and for the girls' divisions of the larger schools are practice in comprehending and representing regular surfaces and solids, the facts about lines and angles, and computing areas and volumes. There will be added to this material, for the boys' division of the larger schools, the facts of equality and congruence of surfaces, the comparison of surfaces, the equality of ratios and similarity, or a part of this material.

The instruction begins with the intuitive consideration of geometric forms the concepts of which are to be clarified and deepened as far as possible by the student's own representation and handwork (forming, drawing, folding, cutting out, cutting up, assembling, estimating, weighing, measuring, computing). These exercises are to grow out of the objects in the pupils' surroundings. The method of proof by measuring and movement (translation, superposition, turning) is to

be preferred to the bare chain of reasoning (the Euclidean). The application of the latter may be entirely omitted.

France. I have not seen recent programmes of instruction for France. But the programmes published by the International Commission on the Teaching of Mathematics and the texts of that time both show the care and time taken to make use of intuitive beginnings, and to develop and fix geometric truths in a long course of geometric drawing.

Present Practice in the United States—Geometry in Arithmetics. Most American children go to elementary schools organized on the eight-four plan in which arithmetic is the mathematics taught in the first eight grades. Even in many, I believe most, schools with department teaching and those organized on the six-six plan the prevailing course of study in arithmetic through the eighth grade has not been reorganized. When the junior high school came along we hailed it as an organization that would allow us to offer in the seventh and eighth grades an amount of mathematics approximating that offered in good European schools. The junior high school has brought changes in administration rather than in courses of study. One acute observer has remarked that the junior high school is interested in everything but scholarship.

The text books must be examined rather than the courses of study, to find what mathematics is being taught in American schools. Mathematics below the high school is determined mainly by the contents of arithmetics written for the first eight grades. What instruction in geometry do they give?

I have examined ten sets of arithmetics to find what they give for grades five, six, seven, and eight that might be called geometry. This does not exhaust the supply of texts. But I believe these give a fair cross section. All of these texts have been published within ten years. These are the findings.

The fifth grade pupil will be taught the following geometric material: measurement of lines, area of a rectangle, volume of a rectangular solid, and drawing to scale. This is about nine per cent of the text.

In the sixth grade eight out of ten of these books review the topics of the fifth grade; nine give the area of a triangle; three give the area of a parallelogram; and two, the area of a trapezoid. There are other topics such as graphs, angles, arcs, pi, and area of a circle given

in one or two texts. The average space given to this material is nine per cent of the text.

In the seventh grade six of the ten texts discuss line, bar, and circle graphs, areas of rectangles, parallelograms, triangles, and trapezoids; four of the books treat all, and five a part of these topics—area of a circle, volume of a rectangular solid, the ordinary constructions with ruler and compasses, parallels, and geometric designs. There are several other topics given in one, two, or three books. The average amount of space given to geometry in the seventh-grade texts is twenty-three per cent.

In the eighth grade all the books give surfaces and volumes of common solids and the Pythagorean theorem; six give similar figures; at least three offer these topics—four or five offer some of them—scale drawing, graphs, and review of areas. A few other topics are given in one or two books. The average amount of space given to geometry in eighth grade texts is nineteen per cent.

What Geometry then do all of these texts offer? They give the conventional mensuration, some graphing, drawing to scale, and the Pythagorean theorem. In five cases this covers fourteen per cent of the space in the books for grades from the fifth to the eighth, and in the other five from fifteen to eighteen per cent. Some of the books offer besides this the common constructions with ruler and compasses, use of protractor, similar figures, sum of the angles of a triangle, designs, use of triangle and T-square in drawing, and a few other topics. There has been a striking improvement in the geometry offered in some of these books compared to that offered a few years ago, but no one of them offers enough to give a fair preparation for demonstrative geometry.

Should not geometry be given more than one-seventh of the time given to mathematics in grades five to eight? I think it should. It gets more time than that in European schools. Not only is it of much practical use, necessary to demonstrative geometry, but in these grades most pupils get all the solid geometry that they ever get in school.

INTRODUCTIONS IN PLANE GEOMETRY TEXTS

I have examined the introductions to plane geometry given in ten texts. Two of them were published a little more than thirty years ago. One of these gives five and the other six pages of definitions as an introduction. Two other texts were published about fifteen years

ago. One of these gives a twelve-page introduction which is purely formal. The other gives thirty-three pages of useful material, as good as some of the quite recent books. The other six texts have been published within the last three years. The number of pages in the introduction varies from twenty-nine to fifty-five, the average being thirty-nine pages. I mention the older geometries only to show that there has been a marked improvement.

To compare and estimate the value of the introductions in the six recent texts, I wrote down nine objectives of an introduction to plane geometry. With this list of objectives I shall give the range of ratings that I gave these geometries with respect to each objective. The scale of rating is zero, very poor, poor, fair, good, very good, and excellent.

OBJECTIVES OF AN INTRODUCTION TO PLANE GEOMETRY

1. Create interest: by pointing out geometric forms in nature, showing the uses of geometry, giving some of the history of geometry, and arousing desires to give proofs. Ratings: Poor to fair.
2. Teach meanings of terms. Ratings: Fair to very good.
3. Give practice in measuring with ruler and compasses. Ratings: Good to very good.
4. Teach construction with ruler and compasses. Ratings: Fair to very good.
5. Give familarity with facts that are to be proved later. Ratings: Fair to good.
6. Give practice in generalization—discovery of theorems. Ratings: Fair to good.
7. Show need for proof. Ratings: Zero to fair.
8. Begin informal proofs and lead to formal proofs. Ratings: Fair to very good.
9. State postulates and give them meaning; organizing intuitions. Ratings: Zero to very poor.

These introductions, in my opinion, should usually be longer and furnish more examples, applications, and thought-provoking questions. They should be organized so as to make the purposes of the work done more apparent. A technique of teaching this material is not so well-known as that of teaching proofs. I believe this material, which should arouse lively interest, is often thought dull by both pupils and teacher who hurry through it because its purpose is not

always clear and it does not lead to well defined and clearly understood results.

The best preparation for demonstrative geometry is given in texts written specially for the seventh and eighth grades of junior high schools. Pupils who master the geometry in these texts should be well prepared to begin demonstrative geometry. But only a small fraction of pupils in public schools use these texts. The preparation for demonstrative geometry of most pupils, therefore, is given in the arithmetics and in the plane geometries. If we have improvement it must come by putting more intuitive geometry into the arithmetics and into the introductions to texts in plane geometry.

Experience with physical objects is of course indispensable to geometry. "If there were no solid objects in nature there would be no geometry" says Poincaré. The primary objects of a course in intuitive geometry is to organize this experience rather than to furnish it. Demonstrative geometry begins with a number of undefined notions and with a group of axioms and postulates. Tenth grade pupils have had a large part of the experience necessary to form these basic space intuitions, notions formed unconsciously and as a result of universal experience with solid objects. But they may be quite unconscious of these generalizations. Few teachers of elementary geometry are conscious of all the assumptions that the experts in the foundations of geometry find in elementary work. For example, Gauss said the definition of a plane surface "as a surface such that if any two points be joined by a straight line the line lies wholly within the surface," contains a postulate or theorem. What is it?

But these notions which are to be used as the bases of formal proofs have never been put into words, certainly not into precise and unequivocal form. Common experiences are to be stated in technical language. The boy who knows that these "corners fit" must be taught to say these "angles are equal," to use an example from Carson. The main purpose of intuitive geometry is to organize experiences of size and form and to lead to generalizations from them. Until these generalizations are easily recognized in their formal dress and until the pupil is trained in their use, a large part of his energy must be put into merely recalling and stating them and little into applying them in new situations.

I hope it has been made clear that I am not advocating a method of measuring and weighing to prove geometric theorems. Far from that. When we construct a material circle, measure its diameter and

circumference and find the ratio, we are dealing with properties of matter of which the round object is made. This is experimental science and not the science of geometry. Geometry is an ideal construction. Geometry "has for its objects certain ideal solids, absolutely rigid" to quote Poincaré. The one excuse for a year of demonstrative geometry in the high school is to teach pupils the method of geometric proof. That method is not the method of experiment. In geometry as a pure science we rigorously prove mathematical truths from given premises. But the demonstration is empty when it is the repetition of words with hazy meanings.

"If equals are substracted from equals the remainders are equal" says the pupil. "What are the equals that are subtracted?" The answer is a blank look. That blank look will sometimes appear no matter what introductory course has been given. But if the axioms are talked over informally, illustrated many times with familiar objects and ideas, the blank look and the sinking feeling that follows failure will come less often.

If I had accepted the results of the experiments in physics that I made in college I would have had conclusive proof that the laws of physics stated in the texts were not so. These laws gave only rough approximations of the results of my experiments. Experiment and intuition do not give certainty. Not every curve has a tangent. But intuition gives "hunches" and experiment may give a test of hypotheses sufficient to direct toward truth. Are not all of us intuitionalists in the beginning? And do not all the beginnings of our science rest on intuition? Logical rigor has a slow growth. Inventiveness in mathematics—may we not fail to develop it by too little appeal to intuition and by limiting our training too early to logic? "Logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention."

It is difficult in a brief paper to guard statements sufficiently so as not to give a false impression of emphasis. Let me repeat that I think the primary reason for studying geometry in the high school is to teach the methods of demonstrative geometry. But I think we often fail in our object by not giving more care to presenting and organizing the experiences without which the practice in logic is barren. I believe that the Geometry Committee could do no greater service than to formulate plans and to stimulate experiments that would bring our practice up to the levels recommended nearly forty years ago.

A One Year Course in Plane and Solid Geometry

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WE ARE HERE to consider curriculum revision of geometry. It is not only fitting but essential that we review the changes that have been made since the origin of geometry in order to meet the needs of those who are to study it; and that we review the aims and objectives of the course.

We frequently see statements from so-called educators, that there has been but little change in geometry from the time of Euclid, and that we are trying to teach mere children what was intended for special groups of superior adults, but this is not the case as we shall see.

From the time that the human race needed to use spacial relations in its practical affairs there must have grown up empirical rules to meet these needs in order to meet their needs in building their pyramids and in the measuring of their land. The Egyptians are credited with having carried this empirical science further than it had been carried by other nations. And thus geometry had its beginning as a practical tool, not a system of logic.

But when this Egyption collection of facts was organized by the Greeks into a science, little if any thought was given to the practical side. It was the logic of the subject that appealed to the early Greek philosophers. When Euclid collected the work of the Greeks and organized it into his Elements, it was for the use of students and thinkers, not mere children. For nearly two thousand years geometry remained a study for the chosen few. Then it came down into our colleges, first as an elective course, and finally as a required course. Then it was that authors began to attempt to simplify some phases of the original Euclid, keeping his order of theorems.

The first notable breaking away from Euclid was that of Legendre in 1794 when he wrote his Elements of Geometry. This was an attempt to simplify the subject through a different organization of the

sequence of theorems. With the exception of England, Legendre's geometry at once became very popular both in Europe and America.

But both Euclid and Legendre cared little for the practical side. The subject was written and studied for its training in logic.

In our early American colleges Euclid was not even required of all college students. Ward's Mathematics was the required mathematics in many of our colleges for many years during the eighteenth century. This book consisted of about 140 pages of arithmetic, 140 pages of algebra, and 75 pages of geometry. As late as 1784 arithmetic in the freshman year, and algebra in the sophomore year constituted the required mathematics of Harvard College.

In the Memoirs of W. E. Chandler (Harvard 1798) he says, "The sophomore year gave us Euclid to measure our strength. Many haulted at the 'pons asinorum.' " As you know this was Book I, Proposition 47, so they hadn't had much of Euclid when they "haulted."

Soon after the appearance of Legendre's geometry, it was translated from the French by David Brewster and displaced Euclid in many of our colleges.

It has been less than a hundred years ago that geometry began to be a required study for college entrance, and it is with this requirement that geometries have been constantly changed to meet the needs of the less mature minds that had to study the subject.

In 1834 Charles Davies, Professor of Mathematics in the Military Academy at West Point "Revised and adapted Brewster's translation of Legendre to the course of mathematical instruction in the United States."

In 1875, the year before his death, Davies wrote his own Elements of Geometry. In this he followed the sequence of propositions of Legendre. In the preface he says, "Geometry must be made a more practical science. To attain this object, without deranging a system so long used, and so generally approved, an Appendix has been prepared and added to Legendre, embracing many Problems of Geometric construction, and many applications of Algebra to Geometry." This appendix, however, contained only 90 such problems and applications. And thus began the first "Curriculum revision," for necessarily the course of study must be the text book itself.

It was from this book, and without a teacher that I got my first taste of geometry. And I got a far greater thrill from the 90 exercises than from the text itself.

In speaking of these 90 problems given in the appendix, Davies says, in the preface, "It would be unjust to those giving instructions, to add to their daily labors, the additional one of finding appropriate solutions to so many difficult problems: hence a Key has been made for the use of Teachers, in which the best methods of constructions and solutions are fully given."

Ten years later, in 1885, J. H. Van Amringe of Columbia College edited a new edition of Davies geometry.

In a note concerning this edition he says:

> The present edition is the result of a careful re-examination of the work, into which have been incorporated such emendations, in the way of greater clearness of expression or of proof, as could be made without altering it in form or substance.
>
> Practical exercises have been placed at the end of several books, and comprise additional theorems, problems, and numerical exercises upon the principles of the Book or Books preceding. They will, it is hoped, be found of service in accustoming students, early in and throughout their course, to make for themselves practical application of geometrical principles.

This marks the beginning of the rapid changes we have made since. This new innovation of exercises at the end of each book became very popular. Many new textbooks came out about this time and for a few years following this. They all followed rather closely the Legendrean order rather than the Euclidian order. Their exercises were of three general types: the theorem, the problem (or constructions), and a few numerical exercises. These books consisted of both plane and solid geometry. One year, usually the 11th grade, was given to plane and a half year to solid, although the half year to solid was much more universal in the West and Middle West than in the East.

During the last 45 or 50 years we have made many changes, if not improvements, in our curriculum in geometry. There is nothing sacred about the theorems of either Euclid of Legendre, nor the sequence of these theorems. The rigor of the logic must depend upon the maturity of the pupil. We are beginning to assume without proof some of those theorems whose proofs are too subtle for a child in the 10th grade; and also some of those whose truths are too apparent to need proof. The pupil gets his first thrill, perhaps, from the proof that the sum of the three angles of a triangle is equal to 180°. The congruence of triangles and the relations of the angles made by the transversal to a pair of parallel lines are too apparent to him to need proof.

About 25 years ago, J. F. Millis and I, in the enthusiasm of youth, attempted to reform the entire teaching of geometry by three striking changes from the texts in use at that time. While we had to keep the accepted theorems to meet college requirements, we re-arranged the sequence, disregarding the Legendrean order followed exclusively at that time. Then since we did not follow the accepted order, there was no reason for keeping the terms Book I, Book II, etc., and hence we divided the book into chapters. While retaining the three types of exercises in general use—the theorem, the problem, and the numerical exercise—we put in about 250 practical exercises, and included a chapter, following similar triangles, on numerical trigonometry which we called trigonometric ratios. The change was too drastic to be popular at that time. Dr. George Bruce Halstead, a rather well known character at that time among mathematical teachers, told me when the book came out that we had an interesting and useful book but that it was *not* geometry—that we should have called it mensuration.

While the order of theorems in our book has not been adopted by other textbook writers, the idea of practical problems and of trigonometry is now quite common.

And now with this brief review of the changes that have been made in the past, we may consider the general topic of this meeting—what further revisions of the curriculum are desirable? Shall we have a First Course in demonstrative geometry to include some of both plane and solid, followed, perhaps, by an elective Second Course?

The answer to this depends upon the aims of a course in geometry. No doubt the study of geometry by the early Greeks was for the pleasure they got from the logic of the subject. When the subject was given as a college study it was mainly for its training in logical thought. As we saw, Davies and others about 50 years ago began to urge that some thought should be given to its practical value.

Today every teacher of geometry would no doubt list the following aims: training in logical thinking; practical uses of the subject; its preparation for other mathematics; and a certain cultural value. Then the question arises as to whether a year of demonstrative geometry consisting wholly of plane or of plane and solid combined will more adequately fulfill these aims.

We attempt today to fulfill the aims of a course in geometry through a study of theorems proved by the author; the theorem type of exer-

cises; problems or constructions depending upon the theorems; practical problems; and numerical problems, the last being only arithmetical or algebraic applications of the geometrical principles that have been established.

It is through the proved theorem and the theorem type of exercise that we teachers hope to develop the logic value. As I have said before, there is nothing sacred about the theorems of Euclid or Legendre, except in so far as they are required by the college board. There is nothing in the argument made by some teachers that we will have to omit so much of plane geometry in a combined course, for we do not now *finish* plane geoetry. With the newer geometry of the triangle, circle, etc., one can go about as far in plane geometry as his I.Q. will permit.

My Juniors in the State Teachers College, who have had a rather stiff course in mathematics for two years, are getting a far greater thrill and much more training in the logic of the subject from a course in Altshiller-Court's College Geometry than they got from high school geometry, and yet they have no thought that they will have finished geometry when they finish this text.

It is for us to consider, then, whether the pupil in the 10th grade can get the values that we now think the course can give, better from one course than the other.

As to the logic value, I see no reason why a combined course might not be as good or even better than a course in plane geometry alone. We all know that many of the theorems are so evident to a child in the 10th grade that he gets no thrill in their proof, hence these might be accepted without proof from the course in intuitive geometry. And the proof of some theorems is too difficult for children of this age to grasp. Hence for the logic value of the subject it may be as well to depend upon those theorems that the child cannot accept intuitively; upon those theorems whose demonstrations he can follow; upon the theorem type of exercise that are closely related; and extend these into similar types of solid geometry, as to devote the year to plane geometry alone.

And now as to the preparatory value: Unlike algebra, geometry is not to any great extent a preparatory subject. Only a small part of geometry is needed in order to understand the mathematics that follows. And as for the preparatory feature, there is nearly as great a need of solid as of plane in future mathematics. And at present solid

geometry is rapidly becoming a dead science. Of those taking the College Board Examination last year about four times as many took plane geometry as solid. This being so, a combined course might give a better preparatory course for future mathematics than that which high school students are now getting from plane alone.

From the standpoint of practical uses of the subject, no strong brief can be made out for either plane or solid. Most of the every-day uses of the subject are now taught in the intuitive geometry of the junior high school, formerly called mensuration in our arithmetic.

The cultural value is a difficult thing to define. It includes knowing something of a science that has been a part of the knowledge of mankind for over 2,000 years, a knowledge of the meaning of a demonstration, and enables one to see and appreciate the geometry in nature, in art, in architecture, in furniture, in jewelry, etc., all about us. Both plane and solid geometry contribute to this, yet plane geometry, perhaps, makes the greater contribution to this phase of the subject.

Considering, then, every phase of our purpose in teaching the subject, it seems to me that there is some advantage in a curriculum for the tenth grade that includes the simple parts of both plane and solid, to be followed by an elective course in the 12th year that aims largely to meet the logic value of the subject.

In order to do this, we can reduce the number of theorems of plane geometry to those that are clearly fundamental in building up a logical sequence; taking for granted those theorems whose truths are evident from experimentation, and from these theorems deduce the theorems whose truths are not so apparent. To further develop the logic value we can include the theorem type of exercise that follows these fundamental theorems. I should also emphasize the so-called practical problem, the numerical problem, and simple constructions in this course.

We could then give enough solid geometry during the latter third of the year to develop some sense of the limits of plane and an appreciation of some phases of geometry carried to three-dimensional space, and some power to interpret three dimensional figures represented on a plane. We could extend phases of plane, as the notion of an angle, to include the dihedral angle, etc. This is done largely in the chapter we now call "Lines and planes in space." To this I would add the computation of areas and volumes of solids.

I have no more specific recommendations than these to make as to

the course, and would hesitate to make them if I had, until I hear the report of the committee, which has made a much more careful study of the question than I have. In general, however, I would increase rather than decrease the present ratio of 2:1—that is, two semesters of plane to one of solid. Perhaps in a one year combined course the ratio would be more nearly 3:1 to bring out the aims stated in this paper.

It is debatable, perhaps, whether we should carry the solid geometry along side by side with plane or not. There is, of course, that one-to-one correspondence between some of the theorems of plane and solid geometry that are very interesting, and such a course carrying the two along parallel could be worked out. It seems to me, however, that considering the immaturity of the child in the tenth grade, such an arrangement will add to the difficulties of the subject rather than simplify them.

While I am entirely open minded about the matter, as I see the problem now, it seems to me that both the logic and the simplicity of the subject can best be attained by devoting about two-thirds of the year to plane, followed by solid, bringing out the one-to-one correspondence of the theorems of solid to those of plane when such occurs.

In conclusion, then, I have presented the problem as I see it. A change will not make master minds out of morons, but I feel that since most pupils are now getting no solid geometry, a one year course in demonstrative geometry to include both plane and solid geometry will tend to increase rather than decrease the logic value, the preparatory value, the practical value, as well as the cultural value of the subject.

BE SURE TO ORDER THE
FIFTH YEARBOOK BEFORE THE
SUPPLY IS EXHAUSTED

Geometry Measures Land

By W. R. RANSOM
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IS ALGEBRA just a game, played with letters? This seems to be the opinion held by many students. Moreover, they are often inattentive to the rules of the game, and willing to take a chance of finishing on the winning side.

Is geometry just a discussion of abstractions, illustrated by drawings on the blackboard and accompanied by reasons taken from a book?

No teacher, however good, will ever wholly overcome these deplorable student attitudes. When I come to a meeting like this, I realize how many able and resourceful teachers are doing wonders with classes in geometry, keeping the subject vital and yet preserving values which arise from its very abstractness.

In the presence of so many skillful and experienced teachers of geometry, I feel some diffidence in offering certain suggestions which may not be new, may not be valuable, and still worse, may be neither, indeed may have already been well tried out and proved worthless.

When I speak I am expected to lay myself open to attack, and to seek to be suggestive, and even provocative, even to expound a too extreme doctrine, in the hope of jarring loose some too well intrenched conservative. But it certainly is not too radical to hark back and remind you that geometry was named for its connection with the measurement of land. But I am going to hazard the suggestion that geometry has become too much an exercise in pure logic, and that more use might be made of a pre-Greek point of view, that the pupils may be forced to make abstractions for themselves, by leading them away from abstractions in the first place.

Euclid lived about 300 B.C. Thales at about 600 B.C. knew about the angles at the base of an isoscles triangle, and many other things in our geometry. In the absence of any earlier information, I will assume that at about 1,000 B.C. what geometry there was, consisted only of a few rules of thumb. And I wish to carry you back to a somewhat earlier time than that, when it was builders and surveyors, and not pedagogs, who handed on what little was known about lines and planes.

There is a tradition that the first surveying was done in the Nile valley, that the annual floods removed the valley landmarks, so that boundaries had to be restored by measuring from inshore monuments. Our modern system of recording ownership of farm land is none too precise, as you will realize if you ever take a town clerk's records and try therefrom to locate the boundaries of a Vermont farm. So I am somewhat skeptical of the Nile tradition. I wonder whether any record shows that there was such definite property in bottom lands that last year's owner could lay claim to a definite piece requiring the services of a surveyor. Nevertheless, it is certain that wherever crops are sown there must be means of discriminating between one man's harvest and another's. So land surveying was an early art.

But I think the science of lines, angles and measurement arose in the first place from the needs of architecture. The builders were students of geometry before real estate was valuable enough to be surveyed. The builder first learned the convenience of the plane surface, its alikeness in all its parts, its relation to straight lines, themselves alike in all their parts, and the mutual applicability of these two forms. The builder in wood discovered the right angle, learned about the equality of opposite sides of a rectangle, and perhaps the equality of the diagonals. When he erected three pillars, he spanned their tops with beams whose length he took from measures between their feet. He knew such geometry as I and my children learned informally when we played with blocks, rectangles with simply commensurable sides. The primitive builder knew nothing of arches and gables. Circles and angles probably did not concern him. Of solid geometry he probably knew only the perpendicularity of a line to every line in the plane through its foot.

The worker in stone came much later and added little to the knowledge possessed by the log artisan. Artists say that his pillars and capitals were in imitation of the tree-user and that the early stone masons admired wood effects and tried to imitate them in stone.

I feel sure that it was the masons that discovered the existence of parallels and some of their properties. These first arose in the experience of the human race from the laying of bricks or of square tiling. The fact that lines which are at right angles to the same line can be continued and remain equally distant can hardly escape the attention of one who lays a stone floor with fitted blocks.

Since the ancients were probably not very precise about their sur-

veying, while the builders attained great precision and skill, it is somewhat surprising that it was the land and not the wall that named our science. In view of my emphasis on the catch phrase "geometry measures land" I will try to justify the land as against the wall. Even if done by such standards as the Vermont farmer employs, surveying soon calls for a greater variety of knowledge than confronts the builder, for the triangle is inevitable in measuring land. Also the land applications are more fundamental and natural to the mind, they involve greater distances, at once arouse the imagination to dealing with abstract lines, and hypothetical planes. To the builder every line and surface must be seen. To the surveyor most of them must be imagined. The building is an object of scrutiny; the land measurements are an object of inner contemplation.

We learn much geometry by handling blocks and by cutting paper, but when we send our imaginations out over the land we begin to think of space and its properties and not of materials with their coordinate shape and color. So I say: "geometry measures land," and pass over all other applications as secondary, and yield to the primitive builder but little of the credit for what the Greeks named for us Geometry.

I am not suggesting that geometry should be begun in the field as surveying. It is not necessary to take a class out of doors, nor is it desirable to do so, for the traveling slows down the thinking. Geometry demands that we imagine our figures. It is better to imagine them out in the fields than to merely see them on the blackboard.

President Elliot, long our foremost educator, told us of his first year in teaching. He took his class over the Harvard yard with surveyor's tools—against all precedent—and from their red and white rods they were called Elliot's Lancers. The experiment was a noble one, but it failed to justify its continuation. The young men were not to be surveyors, and they might as well have exercised their minds imagining the field work as their bodies in doing it.

Consider how one would begin the study of geometry if at first it were regarded as a land measuring proposition.

One would not begin by defining the plane or the straight line. There are such things as *fences* built upon the *natural surface* of the earth. So long as we are not considering solid geometry, these fulfill the most careful requirements of the most abstract postulates of the most scientific modern expositions of geometry. Lines and planes are indefinables anyway, and the attempt to define them otherwise than

by doing something with them can get us nowhere. Hear again what Euclid sayeth: "A point is that which hath no parts and no dimensions"; "A line is length without breadth"; "A straight line is that which lieth evenly between its extreme points." Are these century honored ideas, or any modern substitutes, any better for beginners than a frank discussion which starts with fences, fence posts, and fence corners?

In our lifetime we have seen geometry reach the ultimate stage of abstraction, in which the point, the line, and the plane have lost their physical connotations, and even become to some extent interchangeable. We cannot expect the beginner to start there, and if he is ever to reach the goal of a correct idea of geometry he needs practice in making abstractions for himself. These abstractions must be made from experience, and I urge that fences will prove more satisfactory than chalk marks or the edges of boxes. As for the plane, it will soon be invented by the pupil as a matter of practical necessity, without its being forced on him ex cathedra and ab initio.

The place to begin is the problem of the recovery of landmarks. We have left two pieces of fence and wish to restore the corner. We do not explain that the fences represent lines nor call them vertical planes, nor say what we mean by straightness. It is a fence, a real fence, and the natural abstraction of line will be made the more naturally as its suggestion is avoided by the teacher.

Two posts enable us to relocate a fence just as two fences locate a post. All this is illustrated by lines on the blackboard. But it is not the lines on the blackboard we are talking about, but a beginning of geometry in the pupil's mind.

I must emphasize the difference between the recovery of landmarks and the idea of congruence, an important distinction, not made in a geometry which begins with abstractions.

A triangular field loses one corner, or a part of one fence. The fence can be restored and the field be exactly as before, and the argument is the argument that follows the well-known beginning of a "superposition proof." This is a landmark recovery. But suppose there are two triangular fields for which the fence for one side of each, and the corners at the two ends have been provided alike and in the same order. We now have a question of congruence, and it can be dealt with honestly, for we have something that can be taken up and applied. Either fence will fit either field, and we may assume the

congruence of the fields exactly as all geometers do, but you have not tried to make the pupil think he could move an abstraction about in an abstract plane.

If the parts are arranged in reverse order, the fence argument brings out the real difficulty of the question, which Euclid seems to evade entirely. I believe that even in the case of spherical triangles, which cannot be "turned over" Euclid passes over the difference between actual and symmetrical congruence. Is it not better that the student should realize that the superposition proof is physical rather than abstract?

We may take either the side-angle-side or the angle-side-angle case to argue by superposition. Many modern books do both the same way. Not so Euclid: he puts in the side-angle-side as his 4th proposition and does not get to the other until the 26th. If one of these is to be assumed, and by that I mean "proved by superposition," the other can be proved by a reductio ad absurdum.

Direction is indefinable, as angle is. I suggest that it is better to begin with direction, which is of the earth earthy. Angle is highly theoretical. It is generally misunderstood by beginners who think the length of the sides has something to do with it. A fence corner need not be defined, but when it is put in place so that one side has a certain direction the direction of the other side is fixed also. The abstract idea of angle will arrive just as soon.

Now it is time to consider an angle by means of a pointer swung around a pin on a marked board. The cruder the better, I say. The student who makes his own abstraction will make a better job of it from a rude instrument than by means of a costly over-detailed instrument. It is enough to assure the student that sufficiently accurate instruments are to be had.

Incidentally the sailor measures angles by "points," an unfortunate name for the geometry class, but an excellent unit. There are eight points in a right angle. Bisection is part of our stock in trade, and trisection of angles our bête noir. Yet I quote this from an up-to-date and really excellent book: "On a flat board about 20 inches square, draw a circle of diameter 10 inches. Divide its circumference into 360 equal parts."

The rough protractor divided at most into points or half points is best for the beginner. He should certainly learn about degrees, but

plague take students who later take calculus and cannot believe that there is any legitimate way to measure angles but by degrees.

I will remark again that the triangles drawn upon the board are never to be the triangles we are talking about, but only *maps* of them. Later the idea of similar triangles will emerge. The idea of similar triangles and the pythagorean squares are the most useful of the complicated ideas in geometry. I think similar triangles should be used in a practical way before the proofs about similarity are fully developed. The idea of maps drawn to scale is a crude beginning. One works around from it and comes to it again as a finish of abstract precision. The problem of finding heights and distances by means of scaled drawings is of great importance. The tree, the cliff, and the width of the river should be as much a part of geometry as the right angle.

Parallel lines present an urgent need for abstraction. You have started with clumsy everyday concepts and confronted the student with the practical difficulties of measurement, and lead him to create for his own convenience in thinking about the world certain ideal relationships. Parallels require him to think about distances which cannot lie in our experience. He learns about precisions not to be found in the natural world. He creates ideal figures whose superior simplicity makes them useful. There must be a transition from the crude to the abstract; but is it not the transition rather than the destination that is most important? The circle, the square, the equilateral triangle, and the right angle are not facts of experience. The student must not think of them as pre-existing. He must learn to create them. And if he creates them he must learn the properties of these creations.

Let us raise the question: is there such a thing as a circle? Or are there only paper discs, car wheels, buttons, and such things with wavy edges whose points are approximately distant from an ill determinted hub? I should not treat the circle as we treated the straight line. The circle is not an earth measuring thing. The Greeks knew it was not earthy: Aristotle's followers thought that terrestrial things moved in straight lines and celestial things in circles.

Thus the ball thrown by hand starts off in a straight line, but as it escapes the earthy contact of the hand it tends to move in a circle, and as it approaches the ground again, its circle straightens out and becomes again an earth-mover's straight line. Thus the parabolic tra-

jectory was thought of as a compromise between the initial and final tangents and the circle of curvature at its upper culmination.

I do not know how definitely Aristotle applied this doctrine, but in a general way it illustrates the disposition of the early geometers to treat the circle with respect. No other line was fit for the celestial bodies like stars and planets to move in. The epicycles were the natural recourse when the circular theories proved insufficient. The belief that only circles and compounds of circles could be used reached their climax in the Alphonsine Tables, about 1250. Kepler, in 1600, finding that the planetary orbits were evidently ellipses and not circles, nor epicycles, and challenged as to how heavenly bodies could follow such non-circular and therefore imperfect curves, could offer no better supposition than that an angel guided them.

What I started to say was that when you come to a new topic in geometry this question should be asked: Was this topic originally created by the builder or surveyor, or was it purely theoretical from the start? Then treat it accordingly.

Triangles, squares, lines, angles and all the indefinables come to us by experience. A discussion of the experience should precede the theorizing. That is to say we conduct our argument in terms of fences and the like as long as it does not create a forced situation to do so. When however a topic is reached in which the theoretical aspect is essential, then develop it without pretense of making practical applications. Parallel lines, circles, regular figures, ... are purely theoretical, and should be discussed on a hypothetical basis. There are of course applications, but the theory did not grow out of them.

The student who is not ready to theorize had better not be forced to do so. There ought to be plenty in geometry for a mind at his stage. If this point of view should come to be understood, there would be some realignment of "requirements in geometry." Not being a high school teacher myself, I do not feel the iron heel of entrance requirements, and am perhaps too optimistic in believing that they will be revised from time to time so as to keep abreast of the best progress made by the teachers of geometry.

When the circle is first circumscribed about a triangle, it should be noted that the sides are not proportional to the angles. They are the chords of double the arcs that measure the opposite angles in the triangle. This should be the occasion for inserting a table of chords of

double angles like Ptolemy's. The table had better be to two decimals only, and the angles entered at half-point intervals. There should be some computing with triangles whose sides are in inches and tenths, and some discussion of the inescapable inaccuracies of measurement and computation.

A transition may soon be made to a table of sines. Indeed I think it quite desirable to include in the first course in geometry the use of sines for solving flat triangles, that is earth measuring triangles, and the use of tangents for solving cliff and tree problems. But I do not think it necessary to go any further in developing ideas that become more systematic in the development of trigonometry.

The problem of the incommensurable ought to be approached in a practical manner. This is nothing new that I am saying. But it is of a piece with my fences, and I will put it in.

Do not start with the *theory* of commensurable and incommensurable. But come to it as a matter of actual measurement. First with rulers measure the room to the nearest foot, and see that the particular unit first chosen must be subdivided. Then measure the lengths of pages with a ruler divided to 16ths or possibly 32nds, with a discussion of the 64ths and 128ths. Then measure the page with a metric rule and confine the subsequent discussions to decimal subdivisions.

But I hark back to a previous suggestion that we consider that things do not have exact lengths to be measured. The roughness of the edge sets a limit to the number of decimals of a meter that can be obtained. Show the student that the idea of exact length does not belong to the physical world at all. Is it not better to be honest with the student and let him realize that the pushing of measurement to extreme accuracy reaches a technical limit when applied to real things?

But it is a distinct achievement for a student to realize that the abstract concepts of geometry are quite another matter, that in measuring them there is no reason to suppose that there is any limit to the fineness of the measuring scale.

The appearance of the incommensurable before the Pythagorean Theorem is probably a logical necessity. But the first treatment of incommensurables should be with a marked skeptical attitude as to whether there is any need of it. It should not be forced down the throat of every pupil. Let some whose minds have reached that stage repeat the proof for the equality of ratios when three parallels cut two transversals. But encourage others, if they wish to do so, to

regard the whole process as probably unnecessary. It is merely a precaution in case incommensurability may exist, like a typhoid inoculation when going to an unknown country. Treat the incommensurable with frank skepticism when you first lay it before your class. Do not appear to take it as a necessity foreknown from the beginning.

When the Pythagorean Theorem is reached, then it will come with compelling force that if we create for ourselves squares and equilateral triangles upon a given line segment, then we have created along with them diagonal and altitude—and side, diagonal, and altitude are incommensurable!

The traditional method of taking one segment in the incommensurable case to divide into an integral number of parts economizes logic. But is not the method of measuring both segments with a conventional scale—imaginary, of course, more likely to produce understanding? The traditional method has had a fair trial, and has certainly not produced results that justify it. To call upon the well established idea of a decimally divided scale, and ratios which keep pace while continually taking on new digits representing smaller subdivisions seems to me in line with the idea of using practical means wherever they are of a primitive character.

A sharp contrast between the primitive conceptions and the abstractions the geometer makes from them is what I urge. Certain topics, triangles, similarity, measurement, areas, rectangular grids, ... should be discussed in the most primitive manner possible. Permit the student to make the necessary abstractions, but do not force them upon him. Certain other topics, circles, parallels, incommensurables, ... are highly abstract and should be presented in strong contrast to the primitives. The non-abstract minds should be encouraged to view abstractions with suspicion.

Return to the primitive at all points where the primitive is the natural way. Emphasize the artificial character of those concepts which are the creation of the geometer rather than of the land measurer. Tolerate a skeptical attitude toward abstractions, but invite the student to bring in abstractions which may add to the convenience of treating land-measurement.

I am not sure that all these suggestions will prove valuable, but they are things I have been thinking about, and I lay them out for your consideration, if haply there may be one living grain in it all.

Rebuilding Geometry

By GEORGE W. EVANS
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THREE OR four years ago the National Council published a review of progress in the teaching of mathematics during the preceding twenty-five years. Among the contributions to that end was the reprinting of an address made to the American Mathematical Society in 1902 by its President, E. H. Moore. Most of the recommendations made in that address were reviewed in detail, many with gratifying results to report; but one was omitted. It was this:

> In setting problems in practical mathematics (arithmetical computation or geometrical construction) the teacher should indicate the amount or percentage of error permitted in the final result. If this amount or percentage is chosen conveniently in the different examples, the student will be led to the general notion of closer and closer approximation to a perfectly definite result, and thus in a practical way to the fundamental notions of the theory of limits and of irrational numbers.

In regard to that suggestion of systematic approximation nothing had been done, and even yet there is only a rare beginning. Apparently it seemed an ultra modern point of view, and was regarded as not belonging in the high school field. Well, it is not modern, or foreign to high school geometry. Measurement by systematic approximation is older than Euclid, and forms a fundamental and most important part of his immortal *Elements*.

Nevertheless, if we seize upon Professor Moore's hopeful words as a pretext, and propose to American teachers that the structure of high school geometry should as in Euclid be based upon measurement, and should be used as in Euclid for the careful and sound development of the idea of continuous number, by means of approximate measurement, we can hardly expect a spontaneous welcome. We shall have to argue our case. To obtain a patient hearing, or anything like it, the present year is probably the first opportunity; if we neglect it, it may be the last for a long time. It comes now because we have to consider some radical rearrangement of the subject with a view to including

topics from solid geometry, without enlarging the time now allotted to plane geometry. We must decide what part of our freight we shall throw to the fishes. For the rest, we have to decide whether, in our attempts to imitate Euclid, we have committed ourselves forever to a style of argument pedantic and verbose, and to an order of topics so involved with the vested interests of standard examinations that any change is a cause of panic; these two questions have answers that we should thoughtfully consider.

One of the sources of prolixity in Euclid's work was the cumbrousness of his algebra. Lines and surfaces and solids were not only geometrical entities, they were symbols for number. Numerical statements which we represent by equations were by him embodied in verbal form, referring to lines by letters chosen absolutely at random. His technical vocabulary of proportion was a device for utilizing equations of this sort; with the invention of our modern notation it should at once have become obsolete, but it has stuck obstinately in geometry to the present day. It hardly seems revolutionary to propose that where Euclid had lines that might represent either numbers or any sort of magnitudes, we should have our familiar algebraic symbols to represent either magnitudes or their measurement numbers.

Another difficulty due to our blind emulation of this patron saint of mathematics is the exaggerated shyness with which we approach the subject of measurement. We misunderstand. Euclid's *Elements* are very largely occupied with measurement, not only for explaining convenient rules of computation, but also for the purpose of scientific approximation and the determination of irrational numbers. There is ample evidence that to the Greeks of Euclid's time the idea back of the ratio of two magnitudes was the measurement number of the antecedent in terms of the consequent.

The measurement of a straight line, or of an angle, is in itself a not uninteresting task. Let your pupils know how a base line is measured in the great government surveys, or how the mariner and the astronomer get their accurate angles. Use decimals,—for practical simplicity in computation and for what we shall in a moment see is theoretical simplicity in exposition. These material measurements, such as the surveyor and astronomer make, belong to physics; geometry studies ideal measurements, with objects utterly perfect and bare of irrelevant qualities, and with methods purely theoretical and unhindered by mechanical limitations. The measurements of physics are

always approximate; the measurements of geometry are sometimes unavoidably so.

For example, consider the theoretical measurement, in terms of a foot, of a line four inches long. If we confine ourselves to decimals, we have at every stage two approximations, one an even number and the other odd, differing by 1 in the last decimal place. The two series of approximations, one the series of those too small and the other of those too great, can be carried on until the margin of error is less than any number we may have decided on in advance, however small. This last feature is of fundamental importance; for the purpose of our study we may call such approximations *converging*. The use of decimals is of great advantage here in making the margin of error obvious at every stage of the work.

We have then from the beginning two new technical terms, defined as follows:

The margin of error is the difference between two approximations, one of which is greater and the other less than the measurement number sought.

Converging approximations are those that can be progressively determined by some rule, so that the margin of error will, if we go far enough, be made less than any definite number stated in advance.

Then there is an axiom, and a theorem. The axiom is:

Every magnitude has a measurement number if there is a unit that can, itself or a fraction of it, be exceeded by that magnitude.

The theorem is:

There cannot be two numbers having the same set of converging approximations.

According to that axiom, the diagonal of a square one foot each way has a length in feet. There is no way of expressing that number in figures, though we have an algebraic symbol for it. The theorem assures us that the number represented by that symbol is perfectly definite. These are the fundamental notions of the theory of limits and of irrational numbers,—numbers that the Greeks often spoke of as "inexpressible."

From the beginning we should have in mind the margin of error, and we should point out converging approximations. As for the axiom just mentioned, and the theorem, questions involving them will turn up repeatedly, and you can choose your occasion. In plane geometry there is the measurement of the line, the angle or arc, the rec-

tangle and the circle; and in solid geometry the prism, the pyramid, the cylinder, the cone, and the sphere. You can point out in every case that the approximations are convergent, and leave the Axiom and the Theorem like a medicine to be taken when needed,—though maybe it would be more effective if taken early and often.

Finally, there are in geometry ratios that have class names, as *sine* and *cosine*. Hitherto in geometry classes we have walked carefully around them, putting our feet piously in the footsteps of our elders. We are even advised now to postpone that ancient formula for the third side of an oblique triangle until we come to trigonometry. On the other hand, we may conceivably find it desirable to make the same free use of such ratios as Archimedes did. In developing his indispensable formulas of solid geometry he referred again and again to the functions of half the central angle of the regular polygon. In one very interesting place he describes the cotangent of that angle as "the ratio of the line that subtends half the sides less one to the side of the polygon." Certainly they are valuable geometrical ideas; and there is at least an advantage of brevity in the modern names.

So, if we did find it desirable to rebuild geometry, we can begin by ignoring these three taboos,—of algebra, of measurement, of the names of trigonometrical ratios. There is also one positive suggestion, to make use of the *Curve of Sections* for the convenient phrasing of Cavalieri's Theorem. In this curve the ordinates represent for any solid the areas of sections parallel to the base, and the abscissas the distances of those sections from the base. The area under the curve will then be numerically the same as the volume of the solid. False prophets will tell you that this will not do, because Euclid never represented an area by a line. Don't believe them; and even if you do, get them to tell you what difference it makes to us.

To base our geometry frankly on measurement; to make the same use here as elsewhere of our ordinary algebra; to use sines and cosines where they serve our proper ends; to tackle with confidence the problem of the incommensurable; and finally to accept such practical things as the curve of sections wherever they will contribute to a fuller understanding and to a more general point of view,—all these things will serve not only to condense our subject surprisingly, but to clarify it. Moreover these few easy and valuable innovations will be in direct line with any further mathematical study.

Such innovations involve fundamental changes in the order of

theorems. On that account many American secondary schools will be anxious about the College Entrance Examination Board, and the Board will hesitate. With their examination standards, as has more than once been said, it is much the same as with the protective tariff—occasional misfits are less harmful than frequent or radical change. A change in content can be managed; but a change in sequence, in spite of official assurance to the contrary, is unquestionably dangerous. Candidates that present proofs reversing the logical order of commonly used text-books are and have been and will be penalized, except where the change takes place in large groups and the authorities have been duly notified in advance.

This enforced conservatism is a grave misfortune. Progress in teaching would be greatly promoted by permitting a freely varying logical order in geometry, without permitting vicious logic. That has never yet been done, on any large scale, in written examinations. It may be impossible, but probably is not. It is worth looking into, even if only to be disappointed.

Supposing now that this harmful restraint cannot be removed, or that no attempt to remove it will be countenanced, we can at least ask that a new order of topics be officially recognized,—if we can agree on one order as satisfactory. For this purpose there is now before you a syllabus,* purely tentative of course, which embodies the suggestions that have just been described. It can serve as a basis for discussion, and as a sample of the sort of thing an individual teacher might want to try out with his pupils.

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* NOTE: In the Syllabus, as printed in THE MATHEMATICS TEACHER for February, the following errors should be corrected: In P 17, page 89, for "the product" read "half the product"; and in S 43, page 94, for "line" read "lune."

Geometry as Preparation for College

By W. R. Longley
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In considering the rôle played by geometry in the preparation of candidates for college work, let us first ask if the present training is satisfactory. Does it meet the tests mentioned by Professor Jackson in his statement of the essentials given in The Mathematics Teacher last December? To answer this question I can give the direct testimony of only a comparatively small number of college teachers of mathematics, but the silence of hundreds of others may be taken to mean general assent.

Professor Jackson has said that "Those who believe in the existing courses in geometry at all will agree that the pupil ought to carry away with him: (1) An adequately comprehensive knowledge of geometric ideas, facts, and processes."

The instructors with whom I have talked agree in general that their students possess a knowledge of geometric ideas which is sufficient for the college courses in mathematics. The instructors usually add that only a small amount of such knowledge is really essential and point out a few instances where ignorance of the simplest facts of solid geometry is surprising. But there has been no complaint about deficiency of knowledge of the facts of plane geometry and I believe we are justified in assuming that the preparation in this phase of the subject is quite satisfactory.

The second point in Professor Jackson's statement is "An intimate acquaintance with the nature of deductive reasoning, as applied not only to detached items of argument, but also to the sustained building up of an extensive and coherent logical structure."

If this means an appreciation of the whole logical structure of geometry, from axioms and definitions to last theorems, it is difficult to test either by performance of college work or by entrance examinations. The testimony of the college instructors is entirely negative, and they would certainly complain if they felt dissatisfied. The difficulty of testing this appreciation by entrance examinations is due largely to the fact that we have no standard order of propositions.

The readers of the papers written by the candidates of the College Entrance Examination Board frequently have difficulty in determining whether the candidate has little appreciation of the logical structure of the whole subject of geometry and of the exact position occupied by a particular proposition or whether his appreciation is so keen that he need not waste a word in proving a theorem which rests on a foundation already laid in the order of his development. From the point of view of an examiner a standard order of propositions would be highly desirable. But the evil of stifling initiative on the part of the teacher is probably much greater than the advantage gained and the College Board in preparing its present syllabus definitely disclaimed any intention of prescribing the order of theorems. In Document No. 108, page 4, we find this statement: "Each of the following syllabi has been arranged in what is regarded as a practicable teaching order, but this order is in no wise prescribed. The candidate may base his answers on any sequence of propositions which is logically sound, and may even quote propositions not listed in the syllabus at all, if they have been presented as standard propositions in his course." This wide latitude makes it difficult in many cases to estimate the pupil's appreciation of the beauty of the logical system presented by his teacher. But there is no evidence that such appreciation is lacking on the part of candidates who are successful in gaining admission to college.

I hope that the new courses in geometry will not be too chaotic, that is, that there will not be too many different courses. I am strongly in favor of the proposed courses, but believe that progress should be slow enough to assure success. It may be that the content of the first year course is sufficiently well defined in the minor requirement of the College Board. At least it furnishes a starting point, with its 96 propositions grouped under eight books of Euclid. When we take the inevitable step of discarding the "book" as a means of grouping and of rearranging this material into an organic whole, the number of possibilities is appallingly large. I should approve of an autocratic committee to tell us how it must be done, at least for the first few years.

The third item in Professor Jackson's list is "Familiarity with the independent use of deductive reasoning through the study of substantial originals."

The present requirement of the College Board was designed to increase this familiarity, to develop greater power in meeting new geo-

metric situations, and to minimize the tendency to make geometry a memory exercise. The effect of this requirement and of the Report of the National Committee on the Reorganization of Mathematics in Secondary Education upon the teaching and study of geometry has been good. The shift of emphasis has been in the right direction and has produced a satisfactory effect upon the college students of mathematics. There is, however, some complaint on the part of college instructors about the inability of students to present a logical argument. They can solve a problem but cannot write out a good explanation giving the essential steps in proper order and avoiding unnecessary complications. They are mentally competent but mathematically inarticulate. In this one matter of insistence upon good form in the proofs of the originals we might improve our teaching of geometry.

The fourth point in Professor Jackson's list is "Some facility in the application of geometrical knowledge in the world of experience." This falls outside of the college class room and we should have to look elsewhere for a measure of our success.

So far as preparation for college is concerned we appear to be justified in saying that the teaching of plane geometry is satisfactory. We cannot be so complacent about solid geometry. This is not because the course is inadequate or the teaching poor, but because there is not enough of it. The second paragraph of the College Board pamphlet begins as follows: "To meet the situation created by the gradual disappearance of solid geometry from the schools, the present requirements have been so modified as to permit teachers greater freedom in the treatment of the material. . . . ." The impression is certainly rather widespread that solid geometry is gradually disappearing from the schools. Just how far this disappearance has progressed is difficult to determine. The record of College Board candidates gives some information. During the past ten years the number of candidates examined by the Board has increased by about 48 per cent. The number of books in Mathematics A, Elementary Algebra Complete, has increased from 5,249 in 1920 to 7,731 in 1929, an increase of more than 47 per cent. The percentage increase in Elementary Algebra is practically the same as in the total number of candidates. In Mathematics C, Plane Geometry, the number of books in 1920 was 5,227 and in 1929 was 6,111, an increase of about 17 per cent. In 1929, however, the number of plane geometry books was smaller than at any time during the last seven years. In Mathematics D, Solid Geometry,

the number of books in 1920 was 1,332 and in 1929 was 1,344. During some years of the interval the number of books in solid geometry has been greater, but the net gain is nearly nothing. While these figures do not tell the whole story, it seems quite certain that, judged by the records of the College Board candidates, the study of solid geometry has merely held its own while other work has increased by almost 50 per cent. Relatively this means an appreciable decrease in the study of solid geometry and this has taken place almost entirely since the statement quoted from the College Board pamphlet was written. It seems not unreasonable to suppose that among the hundreds of thousands of high school pupils who do not take the College Board examinations the study of solid geometry has suffered a fate which is probably no better and possibly much worse.

The efforts made by the National Committee and the College Board to correct this situation have so far been unsuccessful. Perhaps five or six years is not long enough for a complete test but it is long enough for us to feel sure that the problem still requires attention.

In adopting its present requirements the College Board did two things with the avowed purpose of helping the situation in solid geometry. The first was the modification of the requirement in Mathematics D, Solid Geometry, and the second was the introduction of the minor requirement involving one year's study of plane and solid geometry.

The modified requirement in solid geometry has not produced any noticeable effect. This may be due to the fact that the intent and the extent of the modification have not been generally understood, or to the fact that nobody who understands and believes in the modified course has taken the trouble to develop its possibilities, or perhaps to the fact that a sufficiently large number of people do not believe it worth while. It is difficult to judge from the examination papers how many teachers have modified their courses to some extent. Whatever modification has been made has not been effective in attracting a larger number of pupils to the study of solid geometry.

We are in no such uncertainty as to the progress made by the new minor requirement. In the six years that examinations have been set in this subject, they have been taken by less than 100 candidates. Some of these candidates obviously did not know enough geometry to pick the right question paper. We can say with certainty that the progress of the minor requirement has been infinitesimal, in the popu-

lar meaning of that word if not in the technical mathematical sense. It seems quite clear that if the object of the minor requirement is to be attained we must build up a new course and not attempt to have an abbreviated course in plane geometry supplemented by a little introduction to solid geometry. The construction of this new course is a man's-size job. Its importance should attract the best ability in the country and its successful introduction into our schools will mark a significant epoch in our mathematical teaching.

Even if it should be possible to build a new course in plane and solid geometry which would conform in content to the present definition of the minor requirement of the College Board, there is involved in the proposal an additional half year course which would carry the pupil as far as he now goes in the separate courses in plane and solid geometry. This additional half year course is not covered by any present definition of the College Board requirement, so that we are faced with the necessity of suggesting to the College Board a further revision of its definitions. Of course such a revision should wait until the exact content of the two courses has been well determined and until a reasonably large number of schools are ready to undertake the teaching.

Those who contemplate assuming the task of formulating and developing the proposed courses in geometry quite naturally wish to have some feeling of assurance that their work will be received favorably by the colleges and by the intermediate agency, the College Entrance Examination Board. I have absolutely no doubt that the colleges will accept the proposed courses in geometry in place of the traditional ones, but it may not be possible to procure in advance official statements to this effect. At Yale, where perhaps the attitude may be typical of colleges in general, the Board of Admissions appears to be opposed to answering hypothetical questions. No candidate has offered Mathematics cd for admission and the Board has not been called upon to decide a specific case. General rules commonly follow experience with individual cases and the Board is not inclined to use its time and energy in taking official action until the occasion demands it. This should certainly not be construed as representing a hostile attitude and I believe no difficulty will be encountered when one or more candidates offer the minor requirement as stated at present or as modified in the future.

The willingness of the College Board to modify its present definition

of the minor requirement and to introduce a new definition covering the third half year of the new course can not be predicted with absolute certainty. But here I feel entirely confident that no obstacle will be met and that we can rely upon the statement in the Annual Report of the Secretary for 1923. This statement is as follows:

"The College Entrance Examination Board endeavors to maintain an open-minded attitude toward any research relating to the principles which underlie its work and toward any changes that are suggested in its organization. It is the policy of the Board to consider no question as closed and no problem as solved if it is regarded as still unsettled by educators in good standing who have given it careful consideration."

This statement of policy should give us confidence that any suggestions from such a body of teachers as is represented by this meeting will receive prompt and careful consideration. I hope that a request to the College Board for revised definitions of requirements in geometry will not be long delayed and that this will present a natural opportunity for consideration of other phases of mathematical preparation for college, particularly the work of the last year.

The present definition of requirements in mathematics was adopted in April, 1923. In algebra, where the greatest change had occurred, examinations were set under both the old and the new requirements for two years so that the present requirements have been in full force for only five years. Algebra is the most fundamental preparatory subject for college mathematics and college teachers have always complained about the training which their students have had. It may not be out of place to remark parenthetically that the results of the efforts of the complaining college instructors are criticized just as severely by the professors in various departments of science and in the engineering schools. Certainly it is desirable that our students should possess greater facility in the use of algebra when they begin the study of calculus and that they should have greater power over the calculus when they enter advanced courses in mathematics, physics, chemistry, and engineering. But I believe that the fault does not lie primarily in our teaching of algebra. The difficulty is in carrying over the knowledge there gained to later work. So far as algebra is concerned the pupils fail to meet the fourth of Professor Jackson's tests. They have too little facility in the world of experience, that world being largely later courses in mathematics and science.

A remedy for this situation has been proposed in so-called courses

in general mathematics. To overcome the criticism that the pupil does not articulate his separate courses, we are to abolish the separate courses. Instruction in separate courses is likened to storing knowledge in pigeon holes. The hopelessness of developing by this method any reasonable degree of efficiency in finding the right mathematical tool when wanted, has led to the attempt to develop courses in general mathematics. I know very little about how such courses have worked in secondary schools. Somewhat similar attempts in college mathematics have not been very successful. It appears that only one new thing can be taught at a time and each new thing must be fairly closely related to what has preceded.

In my opinion a better remedy than general mathematics is cumulative mathematics. We now have cumulative reviews in algebra and other individual subjects, but the idea is not carried out sufficiently from subject to subject. Moreover it is not review in the usual sense of the word that is most effective. There is something about the psychology of review work that is not quite the equivalent of being able to pick out from previous knowledge the correct mathematical tool for the particular job in hand. The review to be most effective should be unconscious by being connected with something new.

In secondary school mathematics the cumulative idea would first manifest itself by using more algebra in geometry. The greater use of the trigonometry of the right triangle is obvious and makes possible the introduction of much valuable material which formerly had to be omitted because we could use nothing but the 30, 45, and 60 degree angles in problems. But I should hope to see also more problems leading to algebraic equations. It is clear that not much time can be spent in taking up new material in our courses in geometry. But this is not new material and only a little time is necessary at occasional intervals to recall to the pupil algebraic processes that were learned a year or two before. The student must do this in college mathematics and if the idea had become natural in preparatory school I think we should hear much less about poor preparation.

The difficulties of college students are often quite unexpected. Usually, however, it is not due to the fact that they never had the required knowledge in algebra but to the fact that they have not thought of it for so long. After passing the course in algebra they disposed of their textbooks and called the whole business finished. If the cumulative idea should go only so far as to induce the pupils to retain their

books and occasionally look up some point, I believe the gain would be considerable.

The possibility of integrating to some extent the separate courses in mathematics applies to the present courses in plane geometry for one year and, for a diminishing number of pupils, in solid geometry for a half year, but more particularly to the new course, which consists of a year's work in geometry, plane and solid, to be followed, in some cases, by a half year's work in advanced geometry. In constructing this new course we start with a comparatively clean slate and it will be easier to weave into it a judicious mixture of algebraic problems than it would be to introduce them into our present courses so hallowed by the tradition of Euclid.

With the new course in geometry in prospect, the first three of the four years of college preparatory work appear to be in good order. But the fourth year still requires our best thought, if we are to get the maximum results from an educational program designed to carry mathematical instruction at least through the first year of college. Too many students come to college and elect, or are persuaded, to take a course in mathematics after having omitted the subject from their schedules for a full year. The result is that these students miss the main point of a liberal education, which includes a brief acquaintance with the mathematical ideas used by the modern scientist. The great majority of them can spend only one year in studying college mathematics and are forced to devote a large proportion of this time to trigonometry and analytic geometry with the result that the amount of calculus they acquire is just enough to leave them completely confused. And yet an introduction to calculus is the goal of the present Freshman course in college mathematics.

In the terminology of the College Entrance Examination Board the courses now open to seniors in a secondary school are Mathematics B, Advanced Algebra; Mathematics D, Solid Geometry; and Mathematics E, Trigonometry. Each of these courses counts one-half unit and they are the only half units defined by the Board. Hence to meet the most common unit requirements of the colleges, the pupil must take two of these three subjects or none.

In the consideration of this offering, trigonometry may be disposed of quickly. It is quite generally satisfactory. Its value from many points of view is readily recognized and no one objects to it unless he objects to mathematics in the large.

The advanced algebra is not so satisfactory. Only in very special cases is it a required subject and comparatively few candidates offer it. In 1920 there were 448 books in advanced algebra and in 1929 there were 293, a decline of 35 per cent. Relative to the total number of candidates the decline is much greater. In the recent revision of definitions an attempt was made to bolster up the subject by the introduction of brief topics, which, it was thought, might create interest for the pupil and allow greater latitude for the teacher. But when this step was taken, there was a definite feeling among some members of the Commission which framed the statement, that we were only marking time until development in the schools might offer a satisfactory substitute.

The solid geometry has been losing ground and there is no evidence that the modified requirement has helped the situation. Under the proposed plan of the new course, Mathematics D, Solid Geometry, would be replaced by a half year course in advanced geometry, but the effect of this step on the last year of preparatory work is impossible to predict.

There are plenty of teachers in secondary schools and colleges who are quite content, or even anxious, to see the advanced algebra and the advanced geometry disappear altogether from our curricula. They believe that better progress can be made in other ways. Some prefer a general review course for the last half-year. Some would like to start the calculus, at least for the stronger students. A proposal that the College Board should define a requirement in calculus was made when the last revision was under way. It was felt, however, that the suggestion was premature and that we lacked experience which would enable us to make a definition of what could reasonably be expected of any considerable number of pupils.

A brief introduction to the calculus is a possibility for a preparatory subject which is fairly certain to receive more consideration in the future. It has the advantage of containing new ideas which are very stimulating to the better students and these ideas can be presented with a minimum of technique. But it has yet to be demonstrated that such a course has much value for any large percentage of high school pupils. It is difficult to see that it has any permanent value for those students who do not follow it with a college course and for those who do continue the subject in college there is the question as to whether the time was well spent.

Another proposal seems so obvious that it is surprising that no one, so far as I know, has made it before. That is a half unit course in analytic or algebraic geometry. The idea occurred to me recently when I happened to pick up a rather ponderous volume entitled "An Introduction to Algebraical Geometry" by A. C. Jones. The book was published in England in 1912 and at that time the author was senior mathematical master at the Bradford Grammar School. He states that the book is the result of twelve years' daily experience in teaching the subject to students in that school. At the end of the preface we find the statement "The author hopes he has produced a book that will not only make the subject interesting to school boys, but will be a valuable companion to which later on the undergraduate will often refer and from which he will not readily depart."

I hope that no one who happens to look at this book will think that I regard it, or that any one else is likely to regard it, as suitable for American school boys. It contains about three times as much analytic geometry as is usually taught to undergraduates in American colleges and some of the topics, such as tangential equations and trilinear coordinates, are regularly presented only in our graduate schools.

But a course in analytic geometry could readily be constructed which would be within the reach of our high school seniors and which would have peculiar advantages for the last half year of a four year cumulative course. As its name implies, analytic geometry is a combination of algebra and geometry. In it we solve geometric problems by algebra and algebraic problems by geometry. The fundamental ideas are all old and yet no one has ever said that analytic geometry is a review of elementary geometry or of algebra. Geometric knowledge is extended from the properties of the circle to properties of the conic sections in general and curves of higher degrees. Algebraic technique is demanded in such a way that its use becomes natural and habitual. Trigonometry is involved necessarily to some extent and, if desirable, can play a considerable part, as is done in the book mentioned above. Determinants are usually avoided but can be introduced to advantage.

This brief glance at the possibilities seems to show that analytic geometry is an ideal study for the last half of the year before entering college. It furnishes a fairly comprehensive, though disguised, review of all earlier mathematics and thus constitutes a fitting climax to a curriculum which should aim to keep constantly before the pupil the essentials of earlier study. It has a distinct value for the non-college

student in this era of graphs and statistics. Finally it clears the way completely for a Freshman course in calculus with time enough to accomplish some of the results which are so insistently demanded by the departments of science as pre-requisites for their work in the sophomore and later years.

To summarize, let us indicate what seems to be the ideal preparation of a student who is to take one or more years of mathematics in college.

The first three years would consist of two years of algebra as defined by the present statement of the requirement of the College Entrance Examination Board and one year of geometry to be defined by a suitable modification of the present minor requirement, including a certain amount of solid geometry and with a little greater stress on problems involving algebra so that the cumulative nature of mathematical knowledge is appreciated.

The first half of the fourth year would consist of trigonometry.

The second half of the fourth year would offer a choice between analytic geometry and an advanced course in synthetic geometry to complete the material now covered in the separate courses in plane and solid geometry but with considerably more stress on problems involving algebra and trigonometry.

To work out the new courses in geometry will require teachers with the initiative to try experiments, the fortitude to abandon ideas which work poorly, and the patience to carry through a multitude of details. To introduce the courses into the schools of the country will require men and women of unusual force, tact, and determination. We look to this organization and the teachers represented by this meeting to produce the personnel to carry out the program which will, we are sure, result in a significant forward step in our educational progress.

The science of geometry is said to have been discovered first by the Egyptians because when the Nile overflowed and all their lands were over-spread with mud, its origin in the divisions of the land by lines and measurements gave the name to the art. And later, being carried further by the keenness of the philosophers, it measured the spaces of the sea, the heavens, and the air.

ISIDORE OF SEVILLE (*c.* 610)

Plato One of the Three Great Athenian Names

PLATO, a fanciful portrait of whom serves as the frontispiece in this issue, began his life work at a time when according to David Eugene Smith* Greece was never in all her history so desperately in need of men of soul. Plato together with Aristotle and Demosthenes are the three great names of Athens.

Cicero once said, "It is reported of Plato that he came into Italy to make himself acquainted with the Pythagoreans, and that when there he made the acquaintance, among others, of Archytas and Timaeus and learned from them all the tenets of the Pythagoreans."

Plato is reported to have traveled in Egypt, in Sicily, and possibly in Asia where, although he must have traded, he also acquired valuable information which was later of use to him. However, it is pretty generally believed that he owes his great appreciation of geometry to the Pythagorean brotherhood. He is reported to have placed above the entrance to his school of philosophy the first known entrance requirement. It read as follows: "Let no one ignorant of geometry enter my door." He is also said to be the author of the well-known statement, "God eternally geometrizes."

Plato was a student of Socrates and of Euclid of Megara, not the famous geometer.

Plato was more interested in arithmetic than in logistic. In his *Republic* he says:

> For the man of war must learn the art of numbers or he will not know how to array his troops; and the philosopher also, because he has to arise out of the sea of change and lay hold of true being, and therefore he must be an arithmetician. . . . . Arithmetic has a very great and elevating effect, compelling the mind to reason about abstract number.

Plato was particularly interested in the mysticism of numbers, but he left nothing definite to indicate what the content or method of teaching was in his school.

Professor Smith** says of him:

> More than any of his predecessors Plato appreciated the scientific possibilities of geometry. . . . . By his teaching he laid the foundations of the science, insisting upon accurate definitions, clear assumptions, and logical proof. His opposition to

* Smith, David Eugene. *The History of Mathematics.* Vol. I, p. 87. Ginn and Company.

** Ibid., p. 90.

the materialists, who saw in geometry only what was immediately useful to the artisan and the mechanic is made clear by Plutarch (1st century) in his *Life of Marcellus*. . . . . That Plato should have the view here indicated is not a cause for surprise. The world's thinkers have always held it. No man ever created a mathematical theory for practical purposes alone. The applications of mathematics have generally been an afterthought.

NEW BOOKS

Solid Geometry. By William W. Strader and Lawrence D. Rhoads. The John C. Winston Company, 1929, VII+170 pages. Price ——.

Another very well written text in solid geometry of a progressive type has appeared. This book is pleasing in appearance, has excellent drawings, is sufficiently rigorous in most respects and has interesting applications to life situations. These features together should bring the book rather wide usage by progressive teachers. The theorems recommended by the National committee are marked N or n according as they are fundamental or subsidiary theorems. Similarly the theorems of the College Entrance Board are marked C and c.

The text is quite traditional in its use of the double headings "Proposition" and "Theorem." The distinctions are valueless, and the book would be more simple if the word "Proposition" were omitted wherever it occurs as a heading and only "Theorem" used.

There seems to be an impression by the authors that the text will be shorter and easier if it has fewer pages. In accordance with this they have unduly crowded their material. For example, pp. 12, 16, 18-23, 28 and 29 give the impression of foreign textbooks. In some cases the theorem, hypothesis and figure are on one page and the proof on the next page. No doubt better organization would improve the book.

In mensuration of curved figures the text has the error common to most high school textbooks. It fails to distinguish between definition and proof. The area of a sphere is not defined as the number of square inches on its surface. A square inch is flat and square inches could never be fitted into a spherical surface. The area or volume of any curved surface must be defined as the limit of the area or volume of a polyhedron of some kind as the number of sides is increased. The theorem then would be to prove that the area or volume of a polyhedron under such conditions has a limit which is expressible in a formula to be used for computation. The distinction is a fine one but nevertheless very important. The postulation of the substitution "Principles" which are a feature of this book, in a sense makes up for the lack of complete definition. The question remains, however, whether or not a clear definition with the omission of the substitution features might not have been more simple as well as more rigorous.

The book is progressive in the apparent postulation of many so-called "Principles" and "Preliminary Theorems." The authors' intention to postulate these is not clear. They seem to

leave such postulation to the teacher. However, even that step is in the line of progress for high school solid geometry.

The use of Cavalieri's theorem as a postulate is a violation of sequence and rigor which is undoubtedly commendable due to the great gain in the omission of the "devil's coffin" theorem. Here unquestionably the gain in clearness and simplicity offsets the loss in rigor due to the postulation of this theorem.

As a whole the book is superior to most solid geometries, since the criticisms of this book would in many cases be even more pertinent to many other texts. Its progressive tendencies combined with its superior drawings and unusually interesting applications give it an excellence that should merit examination and trial by progressively conservative teachers.

H. C. CHRISTOFFERSON
Miami University
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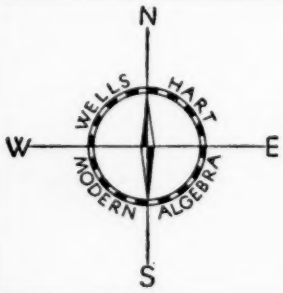

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